# Module 5 - Logarithmic Differentiation

## Introduction

With certain functions containing more complicated products and quotients, differentiation is often made easier if the logarithm of the function is taken before differentiating. This technique, called **'logarithmic differentiation'** is achieved with a knowledge of (i) the laws of logarithms, (ii) the differential coefficients of logarithmic functions, and (iii) the differentiation of implicit functions.

## Laws of Logarithms

**Three laws of logarithms** may be expressed as:

(i) $\log(A \times B) = \log A + \log B$

(ii) $\log\left(\dfrac{A}{B}\right) = \log A - \log B$

(iii) $\log A^n = n \log A$

In calculus, Napierian logarithms (i.e. logarithms to a base of 'e') are invariably used. Thus for two functions $f(x)$ and $g(x)$ the laws of logarithms may be expressed as:

(i) $\ln[f(x) \cdot g(x)] = \ln f(x) + \ln g(x)$

(ii) $\ln\left(\dfrac{f(x)}{g(x)}\right) = \ln f(x) - \ln g(x)$

(iii) $\ln[f(x)]^n = n \ln f(x)$

Taking Napierian logarithms of both sides of the equation $y = \dfrac{f(x) \cdot g(x)}{h(x)}$ gives:

$$\ln y = \ln\left(\frac{f(x) \cdot g(x)}{h(x)}\right)$$

which may be simplified using the above laws of logarithms, giving:

$$\ln y = \ln f(x) + \ln g(x) - \ln h(x)$$

This latter form of the equation is often easier to differentiate.

## A. Differentiation of logarithmic functions

The differential coefficient of the logarithmic function $\ln x$ is given by:

$$\boldsymbol{\frac{d}{dx}(\ln x) = \frac{1}{x}}$$

More generally, it may be shown that:

$$\boldsymbol{\frac{d}{dx}[\ln f(x)] = \frac{f'(x)}{f(x)}} \qquad (1)$$

For example, if $y = \ln(3x^2 + 2x - 1)$ then,

$$\frac{dy}{dx} = \frac{6x + 2}{3x^2 + 2x - 1}$$

Similarly, if $y = \ln(\sin 3x)$ then

$$\frac{dy}{dx} = \frac{3\cos 3x}{\sin 3x} = 3\cot 3x.$$

By using the function of a function rule:

$$\boldsymbol{\frac{d}{dx}(\ln y) = \left(\frac{1}{y}\right)\frac{dy}{dx}} \qquad (2)$$

Differentiation of an expression such as $y = \dfrac{(1+x)^2\sqrt{(x-1)}}{x\sqrt{(x+2)}}$ may be achieved by using the product and quotient rules of differentiation; however the working would be rather complicated. With logarithmic differentiation the following procedure is adopted:

(i) Take Napierian logarithms of both sides of the equation.

Thus $$\ln y = \ln\left\{\frac{(1+x)^2\sqrt{(x-1)}}{x\sqrt{(x+2)}}\right\}$$
$$= \ln\left\{\frac{(1+x)^2(x-1)^{\frac{1}{2}}}{x(x+2)^{\frac{1}{2}}}\right\}$$

(ii) Apply the laws of logarithms.

Thus $\ln y = \ln(1+x)^2 + \ln(x-1)^{\frac{1}{2}}$
$- \ln x - \ln(x+2)^{\frac{1}{2}}$, by laws (i) and (ii)

i.e. $\ln y = 2\ln(1+x) + \frac{1}{2}\ln(x-1)$
$- \ln x - \frac{1}{2}\ln(x+2)$, by law (iii)

(iii) Differentiate each term in turn with respect to $x$ using equations (1) and (2).

Thus $$\frac{1}{y}\frac{dy}{dx} = \frac{2}{(1+x)} + \frac{\frac{1}{2}}{(x-1)} - \frac{1}{x} - \frac{\frac{1}{2}}{(x+2)}$$

(iv) Rearrange the equation to make $\frac{dy}{dx}$ the subject.

Thus $$\frac{dy}{dx} = y\left\{\frac{2}{(1+x)} + \frac{1}{2(x-1)} - \frac{1}{x} - \frac{1}{2(x+2)}\right\}$$

(v) Substitute for $y$ in terms of $x$.

Thus $$\boldsymbol{\frac{dy}{dx} = \frac{(1+x)^2\sqrt{(x-1)}}{x\sqrt{(x+2)}}\left\{\frac{2}{(1+x)} + \frac{1}{2(x-1)} - \frac{1}{x} - \frac{1}{2(x+2)}\right\}}$$

Problem 1. Use logarithmic differentiation to differentiate $y = \frac{(x+1)(x-2)^3}{(x-3)}$

Following the above procedure:

(i) Since $y = \frac{(x+1)(x-2)^3}{(x-3)}$

then $\ln y = \ln\left\{\frac{(x+1)(x-2)^3}{(x-3)}\right\}$

(ii) $\ln y = \ln(x+1) + \ln(x-2)^3 - \ln(x-3)$,

(iii) Differentiating with respect to $x$ gives:
$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{(x+1)} + \frac{3}{(x-2)} - \frac{1}{(x-3)},$$
by using equations (1) and (2)

(iv) Rearranging gives:
$$\frac{dy}{dx} = y\left\{\frac{1}{(x+1)} + \frac{3}{(x-2)} - \frac{1}{(x-3)}\right\}$$

(v) Substituting for $y$ gives:
$$\boldsymbol{\frac{dy}{dx} = \frac{(x+1)(x-2)^3}{(x-3)}\left\{\frac{1}{(x+1)} + \frac{3}{(x-2)} - \frac{1}{(x-3)}\right\}}$$

Problem 2. Differentiate $y = \frac{\sqrt{(x-2)^3}}{(x+1)^2(2x-1)}$ with respect to $x$ and evaluate $\frac{dy}{dx}$ when $x = 3$.

Using logarithmic differentiation and following the above procedure:

(i) Since $y = \frac{\sqrt{(x-2)^3}}{(x+1)^2(2x-1)}$

then $\ln y = \ln\left\{\frac{\sqrt{(x-2)^3}}{(x+1)^2(2x-1)}\right\}$
$= \ln\left\{\frac{(x-2)^{\frac{3}{2}}}{(x+1)^2(2x-1)}\right\}$

(ii) $\ln y = \ln(x-2)^{\frac{3}{2}} - \ln(x+1)^2 - \ln(2x-1)$

i.e. $\ln y = \frac{3}{2}\ln(x-2) - 2\ln(x+1) - \ln(2x-1)$

(iii) $$\frac{1}{y}\frac{dy}{dx} = \frac{\frac{3}{2}}{(x-2)} - \frac{2}{(x+1)} - \frac{2}{(2x-1)}$$

(iv) $$\frac{dy}{dx} = y\left\{\frac{3}{2(x-2)} - \frac{2}{(x+1)} - \frac{2}{(2x-1)}\right\}$$

(v) $$\boldsymbol{\frac{dy}{dx} = \frac{\sqrt{(x-2)^3}}{(x+1)^2(2x-1)}\left\{\frac{3}{2(x-2)} - \frac{2}{(x+1)} - \frac{2}{(2x-1)}\right\}}$$

$$\text{When } x=3,\ \frac{dy}{dx}=\frac{\sqrt{(1)^3}}{(4)^2(5)}\left(\frac{3}{2}-\frac{2}{4}-\frac{2}{5}\right)$$

$$=\pm\frac{1}{80}\left(\frac{3}{5}\right)=\pm\frac{\mathbf{3}}{\mathbf{400}} \text{ or } \pm\mathbf{0.0075}$$

Problem 3. Given $y=\dfrac{3e^{2\theta}\sec 2\theta}{\sqrt{(\theta-2)}}$ determine $\dfrac{dy}{d\theta}$

Using logarithmic differentiation and following the procedure gives:

(i) Since $\quad y=\dfrac{3e^{2\theta}\sec 2\theta}{\sqrt{(\theta-2)}}$

then $\quad \ln y=\ln\left\{\dfrac{3e^{2\theta}\sec 2\theta}{\sqrt{(\theta-2)}}\right\}$

$$=\ln\left\{\frac{3e^{2\theta}\sec 2\theta}{(\theta-2)^{\frac{1}{2}}}\right\}$$

(ii) $\ln y=\ln 3e^{2\theta}+\ln\sec 2\theta-\ln(\theta-2)^{\frac{1}{2}}$

i.e. $\quad \ln y=\ln 3+\ln e^{2\theta}+\ln\sec 2\theta - \frac{1}{2}\ln(\theta-2)$

i.e. $\quad \ln y=\ln 3+2\theta+\ln\sec 2\theta-\frac{1}{2}\ln(\theta-2)$

(iii) Differentiating with respect to $\theta$ gives:

$$\frac{1}{y}\frac{dy}{d\theta}=0+2+\frac{2\sec 2\theta\tan 2\theta}{\sec 2\theta}-\frac{\frac{1}{2}}{(\theta-2)}$$

from equations (1) and (2)

(iv) Rearranging gives:

$$\frac{dy}{d\theta}=y\left\{2+2\tan 2\theta-\frac{1}{2(\theta-2)}\right\}$$

(v) Substituting for $y$ gives:

$$\mathbf{\frac{dy}{d\theta}=\frac{3e^{2\theta}\sec 2\theta}{\sqrt{(\theta-2)}}\left\{2+2\tan 2\theta-\frac{1}{2(\theta-2)}\right\}}$$

Problem 4. Differentiate $y=\dfrac{x^3\ln 2x}{e^x\sin x}$ with respect to $x$.

Using logarithmic differentiation and following the procedure gives:

(i) $\ln y=\ln\left\{\dfrac{x^3\ln 2x}{e^x\sin x}\right\}$

(ii) $\ln y=\ln x^3+\ln(\ln 2x)-\ln(e^x)-\ln(\sin x)$

i.e. $\ln y=3\ln x+\ln(\ln 2x)-x-\ln(\sin x)$

(iii) $\dfrac{1}{y}\dfrac{dy}{dx}=\dfrac{3}{x}+\dfrac{\frac{1}{x}}{\ln 2x}-1-\dfrac{\cos x}{\sin x}$

(iv) $\dfrac{dy}{dx}=y\left\{\dfrac{3}{x}+\dfrac{1}{x\ln 2x}-1-\cot x\right\}$

(v) $\mathbf{\dfrac{dy}{dx}=\dfrac{x^3\ln 2x}{e^x\sin x}\left\{\dfrac{3}{x}+\dfrac{1}{x\ln 2x}-1-\cot x\right\}}$

**Exercise 17. Differentiation of logarithmic functions**

## B. Differentiation of $[f(x)]^x$

Whenever an expression to be differentiated contains a term raised to a power which is itself a function of the variable, then logarithmic differentiation must be used. For example, the differentiation of expressions such as $x^x, (x+2)^x, \sqrt[x]{(x-1)}$ and $x^{3x+2}$ can only be achieved using logarithmic differentiation.

Problem 5. Determine $\dfrac{dy}{dx}$ given $y = x^x$.

Taking Napierian logarithms of both sides of $y = x^x$ gives:

$\ln y = \ln x^x = x \ln x$

Differentiating both sides with respect to $x$ gives:

$\dfrac{1}{y}\dfrac{dy}{dx} = (x)\left(\dfrac{1}{x}\right) + (\ln x)(1)$, using the product rule

i.e. $\quad \dfrac{1}{y}\dfrac{dy}{dx} = 1 + \ln x,$

from which, $\quad \dfrac{dy}{dx} = y(1 + \ln x)$

i.e. $\quad \mathbf{\dfrac{dy}{dx} = x^x(1 + \ln x)}$

Problem 6. Evaluate $\dfrac{dy}{dx}$ when $x = -1$ given $y = (x+2)^x$.

Taking Napierian logarithms of both sides of $y = (x+2)^x$ gives:

$$\ln y = \ln(x+2)^x = x \ln(x+2), \text{ by law (iii)}$$

Differentiating both sides with respect to $x$ gives:

$$\frac{1}{y}\frac{dy}{dx} = (x)\left(\frac{1}{x+2}\right) + [\ln(x+2)](1),$$

by the product rule.

Hence $\quad \dfrac{dy}{dx} = y\left(\dfrac{x}{x+2} + \ln(x+2)\right)$

$$= \mathbf{(x+2)^x\left\{\frac{x}{x+2} + \ln(x+2)\right\}}$$

When $x = -1$, $\quad \mathbf{\dfrac{dy}{dx}} = (1)^{-1}\left(\dfrac{-1}{1} + \ln 1\right)$

$$= (+1)(-1) = \mathbf{-1}$$

Problem 7. Determine (a) the differential coefficient of $y = \sqrt[x]{(x-1)}$ and (b) evaluate $\dfrac{dy}{dx}$ when $x = 2$.

(a) $y = \sqrt[x]{(x-1)} = (x-1)^{\frac{1}{x}}$, since by the laws of indices $\sqrt[n]{a^m} = a^{\frac{m}{n}}$

Taking Napierian logarithms of both sides gives:

$$\ln y = \ln(x-1)^{\frac{1}{x}} = \frac{1}{x}\ln(x-1),$$

Differentiating each side with respect to $x$ gives:

$$\frac{1}{y}\frac{dy}{dx} = \left(\frac{1}{x}\right)\left(\frac{1}{x-1}\right) + [\ln(x-1)]\left(\frac{-1}{x^2}\right),$$

by the product rule.

Hence $\quad \dfrac{dy}{dx} = y\left\{\dfrac{1}{x(x-1)} - \dfrac{\ln(x-1)}{x^2}\right\}$

i.e. $\mathbf{\dfrac{dy}{dx} = \sqrt[x]{(x-1)}\left\{\dfrac{1}{x(x-1)} - \dfrac{\ln(x-1)}{x^2}\right\}}$

(b) When $x = 2$, $\quad \mathbf{\dfrac{dy}{dx}} = \sqrt[2]{(1)}\left\{\dfrac{1}{2(1)} - \dfrac{\ln(1)}{4}\right\}$

$$= \pm 1\left\{\frac{1}{2} - 0\right\} = \mathbf{\pm\frac{1}{2}}$$

Problem 8. Differentiate $x^{3x+2}$ with respect to $x$.

Let $y = x^{3x+2}$
Taking Napierian logarithms of both sides gives:

$$\ln y = \ln x^{3x+2}$$

i.e. $\ln y = (3x+2)\ln x$.
Differentiating each term with respect to $x$ gives:

$$\frac{1}{y}\frac{dy}{dx} = (3x+2)\left(\frac{1}{x}\right) + (\ln x)(3),$$

by the product rule.

Hence $\quad \dfrac{dy}{dx} = y\left\{\dfrac{3x+2}{x} + 3\ln x\right\}$

$$= x^{3x+2}\left\{\frac{3x+2}{x} + 3\ln x\right\}$$

$$= \mathbf{x^{3x+2}\left\{3 + \frac{2}{x} + 3\ln x\right\}}$$

**Exercise 18. Differentiating $[f(x)]^x$ type functions**