Module 9 - Total Differential, Rates of Change & Small Changes

A. Total differential

Partial differentiation is introduced for the case where only one variable changes at a time, the other variables being kept constant. In practice, variables may all be changing at the same time.

If $z = f(u, v, w, \dots)$, then the **total differential**, $\mathbf{d}z$, is given by the sum of the separate partial differentials of z,

i.e. $$\mathbf{d}z = \frac{\partial z}{\partial u}\,\mathbf{d}u + \frac{\partial z}{\partial v}\,\mathbf{d}v + \frac{\partial z}{\partial w}\,\mathbf{d}w + \dots \quad (1)$$

Problem 1. If $z = f(x, y)$ and $z = x^2y^3 + \frac{2x}{y} + 1$, determine the total differential, $\mathrm{d}z$.

The total differential is the sum of the partial differentials,

i.e. $$\mathrm{d}z = \frac{\partial z}{\partial x}\,\mathrm{d}x + \frac{\partial z}{\partial y}\,\mathrm{d}y$$

$$\frac{\partial z}{\partial x} = 2xy^3 + \frac{2}{y} \quad \text{(i.e. } y \text{ is kept constant)}$$

$$\frac{\partial z}{\partial y} = 3x^2y^2\,\frac{2x}{y^2} \quad \text{(i.e. } x \text{ is kept constant)}$$

Hence $$\mathbf{d}z = \left(2xy^3 + \frac{2}{y}\right)\mathbf{d}x + \left(3x^2y^2 - \frac{2x}{y^2}\right)\mathbf{d}y$$

Problem 2. If $z = f(u, v, w)$ and $z = 3u^2 - 2v + 4w^3v^2$ find the total differential, $\mathrm{d}z$.

The total differential

$$\mathrm{d}z = \frac{\partial z}{\partial u}\,\mathrm{d}u + \frac{\partial z}{\partial v}\,\mathrm{d}v + \frac{\partial z}{\partial w}\,\mathrm{d}w$$

$$\frac{\partial z}{\partial u} = 6u \quad \text{(i.e. } v \text{ and } w \text{ are kept constant)}$$

$$\frac{\partial z}{\partial v} = -2 + 8w^3v$$

(i.e. u and w are kept constant)

$$\frac{\partial z}{\partial w} = 12w^2v^2 \quad \text{(i.e. } u \text{ and } v \text{ are kept constant)}$$

Hence

$$\mathbf{d}z = 6u\,\mathbf{d}u + (8vw^3 - 2)\,\mathbf{d}v + (12v^2w^2)\,\mathbf{d}w$$

Problem 3. The pressure p, volume V and temperature T of a gas are related by $pV = kT$, where k is a constant. Determine the total differentials (a) $\mathrm{d}p$ and (b) $\mathrm{d}T$ in terms of p, V and T.

(a) Total differential $\mathrm{d}p = \frac{\partial p}{\partial T}\,\mathrm{d}T + \frac{\partial p}{\partial V}\,\mathrm{d}V$.

Since $pV = kT$ then $p = \frac{kT}{V}$

hence $\frac{\partial p}{\partial T} = \frac{k}{V}$ and $\frac{\partial p}{\partial V} = -\frac{kT}{V^2}$

Thus $\mathrm{d}p = \frac{k}{V}\,\mathrm{d}T - \frac{kT}{V^2}\,\mathrm{d}V$

Since $pV = kT, k = \frac{pV}{T}$

Hence $$\mathrm{d}p = \frac{\left(\frac{pV}{T}\right)}{V}\,\mathrm{d}T - \frac{\left(\frac{pV}{T}\right)T}{V^2}\,\mathrm{d}V$$

i.e. $$\mathbf{d}p = \frac{p}{T}\,\mathbf{d}T - \frac{p}{V}\,\mathbf{d}V$$

(b) Total differential $\mathrm{d}T = \frac{\partial T}{\partial p}\,\mathrm{d}p + \frac{\partial T}{\partial V}\,\mathrm{d}V$

Since $pV = kT, T = \frac{pV}{k}$

hence $\frac{\partial T}{\partial p} = \frac{V}{k}$ and $\frac{\partial T}{\partial V} = \frac{p}{k}$

Thus $\mathrm{d}T = \frac{V}{k}\,\mathrm{d}p + \frac{p}{k}\,\mathrm{d}V$ and substituting $k = \frac{pV}{T}$ gives:

$$\mathrm{d}T = \frac{V}{\left(\frac{pV}{T}\right)}\,\mathrm{d}p + \frac{p}{\left(\frac{pV}{T}\right)}\,\mathrm{d}V$$

i.e. $\mathbf{dT = \frac{T}{p}\,dp + \frac{T}{V}\,dV}$

Exercise 25. Total differential

B. Rates of change

Sometimes it is necessary to solve problems in which different quantities have different rates of change. From equation (1), the rate of change of z, $\frac{\mathrm{d}z}{\mathrm{d}t}$ is given by:

$$\frac{\mathrm{d}z}{\mathrm{d}t} = \frac{\partial z}{\partial u}\frac{\mathrm{d}u}{\mathrm{d}t} + \frac{\partial z}{\partial v}\frac{\mathrm{d}v}{\mathrm{d}t} + \frac{\partial z}{\partial w}\frac{\mathrm{d}w}{\mathrm{d}t} + \cdots \qquad (2)$$

Problem 4. If $z = f(x, y)$ and $z = 2x^3 \sin 2y$ find the rate of change of z, correct to 4 significant figures, when x is 2 units and y is $\pi/6$ radians and when x is increasing at 4 units/s and y is decreasing at 0.5 units/s.

Using equation (2), the rate of change of z,

$$\frac{\mathrm{d}z}{\mathrm{d}t} = \frac{\partial z}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial z}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t}$$

Since $z = 2x^3 \sin 2y$, then

$$\frac{\partial z}{\partial x} = 6x^2 \sin 2y \text{ and } \frac{\partial z}{\partial y} = 4x^3 \cos 2y$$

Since x is increasing at 4 units/s, $\frac{\mathrm{d}x}{\mathrm{d}t} = +4$

and since y is decreasing at 0.5 units/s, $\frac{\mathrm{d}y}{\mathrm{d}t} = -0.5$

Hence $\frac{\mathrm{d}z}{\mathrm{d}t} = (6x^2 \sin 2y)(+4) + (4x^3 \cos 2y)(-0.5)$
$= 24x^2 \sin 2y - 2x^3 \cos 2y$

When $x = 2$ units and $y = \frac{\pi}{6}$ radians, then

$$\frac{\mathrm{d}z}{\mathrm{d}t} = 24(2)^2 \sin[2(\pi/6)] - 2(2)^3 \cos[2(\pi/6)]$$
$$= 83.138 - 8.0$$

Hence the rate of change of z, $\mathbf{\frac{dz}{dt} = 75.14}$ **units/s**, correct to 4 significant figures.

Problem 5. The height of a right circular cone is increasing at 3 mm/s and its radius is decreasing at 2 mm/s. Determine, correct to 3 significant figures, the rate at which the volume is changing (in cm^3/s) when the height is 3.2 cm and the radius is 1.5 cm.

Volume of a right circular cone, $V = \frac{1}{3}\pi r^2 h$
Using equation (2), the rate of change of volume,

$$\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{\partial V}{\partial r}\frac{\mathrm{d}r}{\mathrm{d}t} + \frac{\partial V}{\partial h}\frac{\mathrm{d}h}{\mathrm{d}t}$$

$$\frac{\partial V}{\partial r} = \frac{2}{3}\pi r h \text{ and } \frac{\partial V}{\partial h} = \frac{1}{3}\pi r^2$$

Since the height is increasing at 3 mm/s,

i.e. 0.3 cm/s, then $\frac{dh}{dt} = +0.3$

and since the radius is decreasing at 2 mm/s,

i.e. 0.2 cm/s, then $\frac{dr}{dt} = -0.2$

Hence $\quad \frac{dV}{dt} = \left(\frac{2}{3}\pi rh\right)(-0.2) + \left(\frac{1}{3}\pi r^2\right)(+0.3)$

$$= \frac{-0.4}{3}\pi rh + 0.1\pi r^2$$

However, $\quad h = 3.2$ cm and $r = 1.5$ cm.

Hence $\quad \frac{dV}{dt} = \frac{-0.4}{3}\pi(1.5)(3.2) + (0.1)\pi(1.5)^2$

$$= -2.011 + 0.707 = -1.304\ \text{cm}^3/\text{s}$$

Thus the rate of change of volume is 1.30 cm^3/s decreasing.

Problem 6. The area A of a triangle is given by $A = \frac{1}{2}ac\sin B$, where B is the angle between sides a and c. If a is increasing at 0.4 units/s, c is decreasing at 0.8 units/s and B is increasing at 0.2 units/s, find the rate of change of the area of the triangle, correct to 3 significant figures, when a is 3 units, c is 4 units and B is $\pi/6$ radians.

Using equation (2), the rate of change of area,

$$\frac{dA}{dt} = \frac{\partial A}{\partial a}\frac{da}{dt} + \frac{\partial A}{\partial c}\frac{dc}{dt} + \frac{\partial A}{\partial B}\frac{dB}{dt}$$

Since $\quad A = \frac{1}{2}ac\sin B, \frac{\partial A}{\partial a} = \frac{1}{2}c\sin B,$

$$\frac{\partial A}{\partial c} = \frac{1}{2}a\sin B \text{ and } \frac{\partial A}{\partial B} = \frac{1}{2}ac\cos B$$

$$\frac{da}{dt} = 0.4 \text{ units/s}, \frac{dc}{dt} = -0.8 \text{ units/s}$$

and $\quad \frac{dB}{dt} = 0.2$ units/s

Hence $\frac{dA}{dt} = \left(\frac{1}{2}c\sin B\right)(0.4) + \left(\frac{1}{2}a\sin B\right)(-0.8) + \left(\frac{1}{2}ac\cos B\right)(0.2)$

When $a = 3$, $c = 4$ and $B = \frac{\pi}{6}$ then:

$$\mathbf{\frac{dA}{dt}} = \left(\frac{1}{2}(4)\sin\frac{\pi}{6}\right)(0.4) + \left(\frac{1}{2}(3)\sin\frac{\pi}{6}\right)(-0.8) + \left(\frac{1}{2}(3)(4)\cos\frac{\pi}{6}\right)(0.2)$$

$= 0.4 - 0.6 + 1.039 =$ **0.839 units^2/s**, correct to 3 significant figures.

Problem 7. Determine the rate of increase of diagonal AC of the rectangular solid, shown in Fig. 23, correct to 2 significant figures, if the sides x, y and z increase at 6 mm/s, 5 mm/s and 4 mm/s when these three sides are 5 cm, 4 cm and 3 cm respectively.

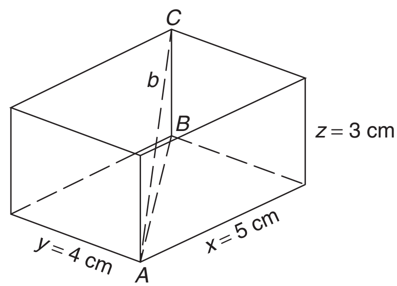

Figure 23

Diagonal $AB = \sqrt{(x^2 + y^2)}$

Diagonal $AC = \sqrt{(BC^2 + AB^2)}$

$$= \sqrt{[z^2 + \{\sqrt{(x^2 + y^2)}\}^2}$$

$$= \sqrt{(z^2 + x^2 + y^2)}$$

Let $AC = b$, then $b = \sqrt{(x^2 + y^2 + z^2)}$

Using equation (2), the rate of change of diagonal b is given by:

$$\frac{db}{dt} = \frac{\partial b}{\partial x}\frac{dx}{dt} + \frac{\partial b}{\partial y}\frac{dy}{dt} + \frac{\partial b}{\partial z}\frac{dz}{dt}$$

Since $b = \sqrt{(x^2 + y^2 + z^2)}$

$$\frac{\partial b}{\partial x} = \frac{1}{2}(x^2 + y^2 + z^2)^{\frac{-1}{2}}(2x) = \frac{x}{\sqrt{(x^2 + y^2 + z^2)}}$$

Similarly, $\frac{\partial b}{\partial y} = \frac{y}{\sqrt{(x^2+y^2+z^2)}}$

and $\frac{\partial b}{\partial z} = \frac{z}{\sqrt{(x^2+y^2+z^2)}}$

$$\frac{dx}{dt} = 6\,\text{mm/s} = 0.6\,\text{cm/s},$$

$$\frac{dy}{dt} = 5\,\text{mm/s} = 0.5\,\text{cm/s},$$

and $\frac{dz}{dt} = 4\,\text{mm/s} = 0.4\,\text{cm/s}$

Hence
$$\frac{db}{dt} = \left[\frac{x}{\sqrt{(x^2+y^2+z^2)}}\right](0.6) + \left[\frac{y}{\sqrt{(x^2+y^2+z^2)}}\right](0.5) + \left[\frac{z}{\sqrt{(x^2+y^2+z^2)}}\right](0.4)$$

When $x = 5$ cm, $y = 4$ cm and $z = 3$ cm, then:

$$\frac{db}{dt} = \left[\frac{5}{\sqrt{(5^2+4^2+3^2)}}\right](0.6) + \left[\frac{4}{\sqrt{(5^2+4^2+3^2)}}\right](0.5) + \left[\frac{3}{\sqrt{(5^2+4^2+3^2)}}\right](0.4)$$

$$= 0.4243 + 0.2828 + 0.1697 = 0.8768\,\text{cm/s}$$

Hence the rate of increase of diagonal AC is 0.88 cm/s or 8.8 mm/s, correct to 2 significant figures.

Exercise 26. Rates of change

C. Small Changes

It is often useful to find an approximate value for the change (or error) of a quantity caused by small changes (or errors) in the variables associated with the quantity. If $z = f(u, v, w, \ldots)$ and $\delta u, \delta v, \delta w, \ldots$ denote **small changes** in $u, v, w, \ldots$ respectively, then the corresponding approximate change δz in z is obtained from equation (1) by replacing the differentials by the small changes.

Thus
$$\boldsymbol{\delta z \approx \frac{\partial z}{\partial u}\delta u + \frac{\partial z}{\partial v}\delta v + \frac{\partial z}{\partial w}\delta w + \cdots} \quad (3)$$

Problem 8. Pressure p and volume V of a gas are connected by the equation $pV^{1.4} = k$. Determine the approximate percentage error in k when the pressure is increased by 4% and the volume is decreased by 1.5%.

Using equation (3), the approximate error in k,

$$\delta k \approx \frac{\partial k}{\partial p}\delta p + \frac{\partial k}{\partial V}\delta V$$

Let p, V and k refer to the initial values.

Since $\quad k = pV^{1.4}$ then $\dfrac{\partial k}{\partial p} = V^{1.4}$

and $\quad \dfrac{\partial k}{\partial V} = 1.4pV^{0.4}$

Since the pressure is increased by 4%, the change in pressure $\delta p = \dfrac{4}{100} \times p = 0.04p$.

Since the volume is decreased by 1.5%, the change in volume $\delta V = \dfrac{-1.5}{100} \times V = -0.015V$.

Hence the approximate error in k,

$$\begin{aligned}\delta k &\approx (V)^{1.4}(0.04p) + (1.4pV^{0.4})(-0.015V)\\ &\approx pV^{1.4}[0.04 - 1.4(0.015)]\\ &\approx pV^{1.4}[0.019] \approx \frac{1.9}{100}pV^{1.4} \approx \frac{1.9}{100}k\end{aligned}$$

i.e. **the approximate error in k is a 1.9% increase.**

Problem 9. Modulus of rigidity $G = (R^4\theta)/L$, where R is the radius, θ the angle of twist and L the length. Determine the approximate percentage error in G when R is increased by 2%, θ is reduced by 5% and L is increased by 4%.

Using $\quad \delta G \approx \dfrac{\partial G}{\partial R}\delta R + \dfrac{\partial G}{\partial \theta}\delta\theta + \dfrac{\partial G}{\partial L}\delta L$

Since $\quad G = \dfrac{R^4\theta}{L}, \dfrac{\partial G}{\partial R} = \dfrac{4R^3\theta}{L}, \dfrac{\partial G}{\partial \theta} = \dfrac{R^4}{L}$

and $\quad \dfrac{\partial G}{\partial L} = \dfrac{-R^4\theta}{L^2}$

Since R is increased by 2%, $\delta R = \dfrac{2}{100}R = 0.02R$

Similarly, $\delta\theta = -0.05\theta$ and $\delta L = 0.04L$

$$\text{Hence } \delta G \approx \left(\frac{4R^3\theta}{L}\right)(0.02R) + \left(\frac{R^4}{L}\right)(-0.05\theta) + \left(-\frac{R^4\theta}{L^2}\right)(0.04L)$$

$$\approx \frac{R^4\theta}{L}[0.08 - 0.05 - 0.04] \approx -0.01\frac{R^4\theta}{L},$$

i.e. $\delta G \approx -\dfrac{1}{100}G$

Hence the approximate percentage error in G is a 1% decrease.

Problem 10. The second moment of area of a rectangle is given by $I = (bl^3)/3$. If b and l are measured as 40 mm and 90 mm respectively and the measurement errors are -5 mm in b and $+8$ mm in l, find the approximate error in the calculated value of I.

Using equation (3), the approximate error in I,

$$\delta I \approx \frac{\partial I}{\partial b}\delta b + \frac{\partial I}{\partial l}\delta l$$

$$\frac{\partial I}{\partial b} = \frac{l^3}{3} \text{ and } \frac{\partial I}{\partial l} = \frac{3bl^2}{3} = bl^2$$

$$\delta b = -5\text{ mm and } \delta l = +8\text{ mm}$$

Hence $\delta I \approx \left(\dfrac{l^3}{3}\right)(-5) + (bl^2)(+8)$

Since $b = 40$ mm and $l = 90$ mm then

$$\begin{aligned}\delta I &\approx \left(\frac{90^3}{3}\right)(-5) + 40(90)^2(8)\\ &\approx -1215000 + 2592000\\ &\approx 1377000\text{ mm}^4 \approx 137.7\text{ cm}^4\end{aligned}$$

Hence the approximate error in the calculated value of I is a 137.7 cm^4 increase.

Problem 11. The time of oscillation t of a pendulum is given by $t = 2\pi\sqrt{\dfrac{l}{g}}$. Determine the approximate percentage error in t when l has an error of 0.2% too large and g 0.1% too small.

Using equation (3), the approximate change in t,

$$\delta t \approx \frac{\partial t}{\partial l}\delta l + \frac{\partial t}{\partial g}\delta g$$

Since $\quad t = 2\pi\sqrt{\dfrac{l}{g}}, \dfrac{\partial t}{\partial l} = \dfrac{\pi}{\sqrt{lg}}$

and $\dfrac{\partial t}{\partial g} = -\pi\sqrt{\dfrac{l}{g^3}}$

$$\delta l = \frac{0.2}{100} l = 0.002\, l \;\text{ and }\; \delta g = -0.001 g$$

hence $\delta t \approx \dfrac{\pi}{\sqrt{lg}}(0.002l) + -\pi\sqrt{\dfrac{l}{g^3}}(-0.001\, g)$

$$\approx 0.002\pi\sqrt{\frac{l}{g}} + 0.001\pi\sqrt{\frac{l}{g}}$$

$$\approx (0.001)\left[2\pi\sqrt{\frac{l}{g}}\right] + 0.0005\left[2\pi\sqrt{\frac{l}{g}}\;\right]$$

$$\approx 0.0015t \approx \frac{0.15}{100} t$$

Hence the approximate error in t is a 0.15% increase.

Exercise 27. Small changes