Module 9
Integration using Trigonometric Substitutions

A. Introduction

Table 2 gives a summary of the integrals that require the use of **trigonometric substitutions**, and their application is demonstrated in Problems 1 to 19.

Solved problems on integration of $\sin^2 x$, $\cos^2 x$, $\tan^2 x$ and $\cot^2 x$

Problem 1. Evaluate: $\int_0^{\frac{\pi}{4}} 2\cos^2 4t\,dt$

Since $\cos 2t = 2\cos^2 t - 1$

then $\cos^2 t = \frac{1}{2}(1+\cos 2t)$ and

$\cos^2 4t = \frac{1}{2}(1+\cos 8t)$

Hence $\int_0^{\frac{\pi}{4}} 2\cos^2 4t\,dt$

$$= 2\int_0^{\frac{\pi}{4}} \frac{1}{2}(1+\cos 8t)\,dt = \left[t + \frac{\sin 8t}{8}\right]_0^{\frac{\pi}{4}}$$

$$= \left[\frac{\pi}{4} + \frac{\sin 8\left(\frac{\pi}{4}\right)}{8}\right] - \left[0 + \frac{\sin 0}{8}\right]$$

$$= \frac{\pi}{4} \quad \text{or} \quad \mathbf{0.7854}$$

Problem 2. Determine: $\int \sin^2 3x\,dx$

Since $\cos 2x = 1 - 2\sin^2 x$

then $\sin^2 x = \frac{1}{2}(1-\cos 2x)$ and

$\sin^2 3x = \frac{1}{2}(1-\cos 6x)$

Hence $\int \sin^2 3x\,dx = \int \frac{1}{2}(1-\cos 6x)\,dx$

$$= \frac{1}{2}\left(x - \frac{\sin 6x}{6}\right) + c$$

Problem 3. Find: $3\int \tan^2 4x\,dx$

Since $1+\tan^2 x = \sec^2 x$, then $\tan^2 x = \sec^2 x - 1$ and $\tan^2 4x = \sec^2 4x - 1$

Hence $3\int \tan^2 4x\,dx = 3\int (\sec^2 4x - 1)\,dx$

$$= 3\left(\frac{\tan 4x}{4} - x\right) + c$$

Problem 4. Evaluate $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{2}\cot^2 2\theta\,d\theta$

Table 2 Integrals using trigonometric substitutions

$f(x)$	$\int f(x)dx$	Method	See problem
1. $\cos^2 x$	$\frac{1}{2}\left(x+\frac{\sin 2x}{2}\right)+c$	Use $\cos 2x = 2\cos^2 x - 1$	1
2. $\sin^2 x$	$\frac{1}{2}\left(x-\frac{\sin 2x}{2}\right)+c$	Use $\cos 2x = 1 - 2\sin^2 x$	2
3. $\tan^2 x$	$\tan x - x + c$	Use $1+\tan^2 x = \sec^2 x$	3
4. $\cot^2 x$	$-\cot x - x + c$	Use $\cot^2 x + 1 = \text{cosec}^2 x$	4
5. $\cos^m x \sin^n x$	(a) If either m or n is odd (but not both), use $\cos^2 x + \sin^2 x = 1$		5, 6
	(b) If both m and n are even, use either $\cos 2x = 2\cos^2 x - 1$ or $\cos 2x = 1 - 2\sin^2 x$		7, 8
6. $\sin A \cos B$		Use $\frac{1}{2}[\sin(A+B)+\sin(A-B)]$	9
7. $\cos A \sin B$		Use $\frac{1}{2}[\sin(A+B)-\sin(A-B)]$	10
8. $\cos A \cos B$		Use $\frac{1}{2}[\cos(A+B)+\cos(A-B)]$	11
9. $\sin A \sin B$		Use $-\frac{1}{2}[\cos(A+B)-\cos(A-B)]$	12
10. $\frac{1}{\sqrt{a^2-x^2}}$	$\sin^{-1}\frac{x}{a}+c$	Use $x = a\sin\theta$ substitution	13, 14
11. $\sqrt{a^2-x^2}$	$\frac{a^2}{2}\sin^{-1}\frac{x}{a}+\frac{x}{2}\sqrt{a^2-x^2}+c$	Use $x = a\sin\theta$ substitution	15, 16
12. $\frac{1}{a^2+x^2}$	$\frac{1}{a}\tan^{-1}\frac{x}{a}+c$	Use $x = a\tan\theta$ substitution	17–19

Since $\cot^2\theta + 1 = \text{cosec}^2\theta$, then

$\cot^2\theta = \text{cosec}^2\theta - 1$ and $\cot^2 2\theta = \text{cosec}^2 2\theta - 1$

Hence $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{2}\cot^2 2\theta \, d\theta$

$$= \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} (\text{cosec}^2 2\theta - 1)\, d\theta = \frac{1}{2}\left[\frac{-\cot 2\theta}{2} - \theta\right]_{\frac{\pi}{6}}^{\frac{\pi}{3}}$$

$$= \frac{1}{2}\left[\left(\frac{-\cot 2\left(\frac{\pi}{3}\right)}{2} - \frac{\pi}{3}\right) - \left(\frac{-\cot 2\left(\frac{\pi}{6}\right)}{2} - \frac{\pi}{6}\right)\right]$$

$$= \frac{1}{2}[(--0.2887 - 1.0472) - (-0.2887 - 0.5236)]$$

$$= \mathbf{0.0269}$$

Exercise 5. Integration of $\sin^2 x$, $\cos^2 x$, $\tan^2 x$ and $\cot^2 x$

Solved problems on powers of sines and cosines

Problem 5. Determine: $\int \sin^5\theta\, d\theta$

Since $\cos^2\theta + \sin^2\theta = 1$ then $\sin^2\theta = (1-\cos^2\theta)$.

Hence $\int \sin^5\theta\, d\theta$

$$= \int \sin\theta(\sin^2\theta)^2\, d\theta = \int \sin\theta(1-\cos^2\theta)^2 d\theta$$

$$= \int \sin\theta(1-2\cos^2\theta+\cos^4\theta)\, d\theta$$

$$= \int (\sin\theta - 2\sin\theta\cos^2\theta + \sin\theta\cos^4\theta) d\theta$$

$$= \mathbf{-\cos\theta + \frac{2\cos^3\theta}{3} - \frac{\cos^5\theta}{5} + c}$$

[Whenever a power of a cosine is multiplied by a sine of power 1, or vice-versa, the integral may be determined by inspection as shown.

In general, $\int \cos^n\theta \sin\theta\, d\theta = \dfrac{-\cos^{n+1}\theta}{(n+1)} + c$

and $\int \sin^n\theta \cos\theta\, d\theta = \dfrac{\sin^{n+1}\theta}{(n+1)} + c$

Problem 6. Evaluate: $\int_0^{\frac{\pi}{2}} \sin^2 x \cos^3 x\, dx$

$$\int_0^{\frac{\pi}{2}} \sin^2 x\cos^3 x\, dx = \int_0^{\frac{\pi}{2}} \sin^2 x\cos^2 x\cos x\, dx$$

$$= \int_0^{\frac{\pi}{2}} (\sin^2 x)(1-\sin^2 x)(\cos x)\, dx$$

$$= \int_0^{\frac{\pi}{2}} (\sin^2 x\cos x - \sin^4 x\cos x)\, dx$$

$$= \left[\frac{\sin^3 x}{3} - \frac{\sin^5 x}{5}\right]_0^{\frac{\pi}{2}}$$

$$= \left[\frac{\left(\sin\frac{\pi}{2}\right)^3}{3} - \frac{\left(\sin\frac{\pi}{2}\right)^5}{5}\right] - [0-0]$$

$$= \frac{1}{3} - \frac{1}{5} = \mathbf{\frac{2}{15}} \text{ or } \mathbf{0.1333}$$

Problem 7. Evaluate: $4\int_0^{\frac{\pi}{4}} 4\cos^4\theta\, d\theta$, correct to significant figures

$$\int_0^{\frac{\pi}{4}} 4\cos^4\theta\, d\theta = 4\int_0^{\frac{\pi}{4}} (\cos^2\theta)^2 d\theta$$

$$= 4\int_0^{\frac{\pi}{4}} \left[\frac{1}{2}(1+\cos 2\theta)\right]^2 d\theta$$

$$= \int_0^{\frac{\pi}{4}} (1+2\cos 2\theta+\cos^2 2\theta)\, d\theta$$

$$= \int_0^{\frac{\pi}{4}} \left[1+2\cos 2\theta + \frac{1}{2}(1+\cos 4\theta)\right] d\theta$$

$$= \int_0^{\frac{\pi}{4}} \left[\frac{3}{2} + 2\cos 2\theta + \frac{1}{2}\cos 4\theta\right] d\theta$$

$$= \left[\frac{3\theta}{2} + \sin 2\theta + \frac{\sin 4\theta}{8}\right]_0^{\frac{\pi}{4}}$$

$$= \left[\frac{3}{2}\left(\frac{\pi}{4}\right) + \sin\frac{2\pi}{4} + \frac{\sin 4(\pi/4)}{8}\right] - [0]$$

$$= \frac{3\pi}{8} + 1$$

$= \mathbf{2.178}$, correct to 4 significant figures.

Problem 8. Find: $\int \sin^2 t \cos^4 t\, dt$

$$\int \sin^2 t \cos^4 t\, dt = \int \sin^2 t (\cos^2 t)^2\, dt$$

$$= \int \left(\frac{1-\cos 2t}{2}\right)\left(\frac{1+\cos 2t}{2}\right)^2 dt$$

$$= \frac{1}{8}\int (1-\cos 2t)(1+2\cos 2t+\cos^2 2t)\, dt$$

$$= \frac{1}{8}\int (1+2\cos 2t+\cos^2 2t-\cos 2t - 2\cos^2 2t - \cos^3 2t)\, dt$$

$$= \frac{1}{8}\int (1+\cos 2t-\cos^2 2t-\cos^3 2t)\, dt$$

$$= \frac{1}{8}\int \left[1+\cos 2t - \left(\frac{1+\cos 4t}{2}\right) - \cos 2t(1-\sin^2 2t)\right] dt$$

$$= \frac{1}{8}\int \left(\frac{1}{2} - \frac{\cos 4t}{2} + \cos 2t \sin^2 2t\right) dt$$

$$= \mathbf{\frac{1}{8}\left(\frac{t}{2} - \frac{\sin 4t}{8} + \frac{\sin^3 2t}{6}\right) + c}$$

Solved problems on integration of products of sines and cosines

Problem 9. Determine: $\int \sin 3t \cos 2t\, dt$

$$\int \sin 3t \cos 2t\, dt$$

$$= \int \frac{1}{2}[\sin(3t+2t)+\sin(3t-2t)]\, dt,$$

from 6 of Table 2,

$$= \frac{1}{2}\int (\sin 5t + \sin t)\, dt$$

$$= \mathbf{\frac{1}{2}\left(\frac{-\cos 5t}{5} - \cos t\right) + c}$$

Problem 10. Find: $\int \frac{1}{3}\cos 5x \sin 2x\, dx$

$$\int \frac{1}{3}\cos 5x \sin 2x\, dx$$

$$= \frac{1}{3}\int \frac{1}{2}[\sin(5x+2x)-\sin(5x-2x)]\, dx,$$

from 7 of Table 2

$$= \frac{1}{6}\int (\sin 7x - \sin 3x)\, dx$$

$$= \mathbf{\frac{1}{6}\left(\frac{-\cos 7x}{7} + \frac{\cos 3x}{3}\right) + c}$$

Exercise 6. Integration of powers of sines and cosines

Problem 11. Evaluate: $\int_0^1 2\cos 6\theta \cos\theta \, d\theta$, correct to 4 decimal places

$$\int_0^1 2\cos 6\theta \cos\theta \, d\theta$$

$$= 2\int_0^1 \frac{1}{2}[\cos(6\theta+\theta)+\cos(6\theta-\theta)]\,d\theta,$$

from 8 of Table 2

$$= \int_0^1 (\cos 7\theta + \cos 5\theta)\,d\theta = \left[\frac{\sin 7\theta}{7}+\frac{\sin 5\theta}{5}\right]_0^1$$

$$= \left(\frac{\sin 7}{7}+\frac{\sin 5}{5}\right)-\left(\frac{\sin 0}{7}+\frac{\sin 0}{5}\right)$$

'sin 7' means 'the sine of 7 radians' ($\equiv 401.07°$) and sin 5 $\equiv$ 286.48°.

Hence $\int_0^1 2\cos 6\theta \cos\theta \, d\theta$

$$= (0.09386 + -0.19178) - (0)$$

$$= \mathbf{-0.0979}, \text{ correct to 4 decimal places}$$

Problem 12. Find: $3\int \sin 5x \sin 3x \, dx$

$$3\int \sin 5x \sin 3x \, dx$$

$$= 3\int -\frac{1}{2}[\cos(5x+3x)-\cos(5x-3x)]\,dx,$$

from 9 of Table 2

$$= -\frac{3}{2}\int (\cos 8x - \cos 2x)\,dx$$

$$= \mathbf{-\frac{3}{2}\left(\frac{\sin 8x}{8}-\frac{\sin 2x}{2}\right)+c} \quad \text{or}$$

$$\mathbf{\frac{3}{16}(4\sin 2x - \sin 8x)+c}$$

Exercise 6. Integration of products of sines and cosines

Solved problems on integration using the sinθ substitution

Problem 13. Determine: $\int \frac{1}{\sqrt{a^2-x^2}}\,dx$

Let $x = a\sin\theta$, then $\frac{dx}{d\theta} = a\cos\theta$ and $dx = a\cos\theta\,d\theta$.

Hence $\int \frac{1}{\sqrt{a^2-x^2}}\,dx$

$$= \int \frac{1}{\sqrt{a^2-a^2\sin^2\theta}}\,a\cos\theta\,d\theta$$

$$= \int \frac{a\cos\theta\,d\theta}{\sqrt{a^2(1-\sin^2\theta)}}$$

$$= \int \frac{a\cos\theta\,d\theta}{\sqrt{a^2\cos^2\theta}}, \text{ since } \sin^2\theta+\cos^2\theta = 1$$

$$= \int \frac{a\cos\theta\,d\theta}{a\cos\theta} = \int d\theta = \theta + c$$

Since $x = a\sin\theta$, then $\sin\theta = \frac{x}{a}$ and $\theta = \sin^{-1}\frac{x}{a}$

Hence $\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \mathbf{\sin^{-1}\frac{x}{a}+c}$

Problem 14. Evaluate: $\int_0^3 \frac{1}{\sqrt{9-x^2}}\,dx$

From Problem 13, $\int_0^3 \frac{1}{\sqrt{9-x^2}}\,dx$

$$= \left[\sin^{-1}\frac{x}{3}\right]_0^3 \quad \text{since } a = 3$$

$$= (\sin^{-1}1 - \sin^{-1}0) = \boldsymbol{\frac{\pi}{2}} \text{ or } \mathbf{1.5708}$$

Problem 15. Find: $\int \sqrt{a^2-x^2}\,dx$

Let $x = a\sin\theta$ then $\frac{dx}{d\theta} = a\cos\theta$ and $dx = a\cos\theta\,d\theta$

Hence $\int \sqrt{a^2-x^2}dx$

$$= \int \sqrt{a^2 - a^2\sin^2\theta}\,(a\cos\theta\,d\theta)$$

$$= \int \sqrt{a^2(1-\sin^2\theta)}\,(a\cos\theta\,d\theta)$$

$$= \int \sqrt{a^2\cos^2\theta}\,(a\cos\theta\,d\theta)$$

$$= \int (a\cos\theta)\,(a\cos\theta\,d\theta)$$

$$= a^2\int \cos^2\theta\,d\theta = a^2\int\left(\frac{1+\cos 2\theta}{2}\right)d\theta$$

$$(\text{since } \cos 2\theta = 2\cos^2\theta - 1)$$

$$= \frac{a^2}{2}\left(\theta + \frac{\sin 2\theta}{2}\right) + c$$

$$= \frac{a^2}{2}\left(\theta + \frac{2\sin\theta\cos\theta}{2}\right) + c$$

since $\sin 2\theta = 2\sin\theta\cos\theta$

$$= \frac{a^2}{2}[\theta + \sin\theta\cos\theta] + c$$

Since $x = a\sin\theta$, then $\sin\theta = \frac{x}{a}$ and $\theta = \sin^{-1}\frac{x}{a}$

Also, $\cos^2\theta + \sin^2\theta = 1$, from which,

$$\cos\theta = \sqrt{1-\sin^2\theta} = \sqrt{1-\left(\frac{x}{a}\right)^2}$$

$$= \sqrt{\frac{a^2-x^2}{a^2}} = \frac{\sqrt{a^2-x^2}}{a}$$

Thus $\int \sqrt{a^2-x^2}\,dx = \frac{a^2}{2}[\theta + \sin\theta\cos\theta]$

$$= \frac{a^2}{2}\left[\sin^{-1}\frac{x}{a} + \left(\frac{x}{a}\right)\frac{\sqrt{a^2-x^2}}{a}\right] + c$$

$$= \boldsymbol{\frac{a^2}{2}\sin^{-1}\frac{x}{a} + \frac{x}{2}\sqrt{a^2-x^2} + c}$$

Problem 16. Evaluate: $\int_0^4 \sqrt{16-x^2}\,dx$

From Problem 15, $\int_0^4 \sqrt{16-x^2}\,dx$

$$= \left[\frac{16}{2}\sin^{-1}\frac{x}{4} + \frac{x}{2}\sqrt{16-x^2}\right]_0^4$$

$$= \left[8\sin^{-1}1 + 2\sqrt{0}\right] - \left[8\sin^{-1}0 + 0\right]$$

$$= 8\sin^{-1}1 = 8\left(\frac{\pi}{2}\right)$$

$$= \mathbf{4\pi} \text{ or } \mathbf{12.57}$$

Exercise 7. integration using the sine θ substitution

Solved problems on integration using the tan θ substitution

Problem 17. Determine: $\int \frac{1}{(a^2+x^2)} dx$

Let $x = a \tan\theta$ then $\frac{dx}{d\theta} = a\sec^2\theta$ and $dx = a\sec^2\theta\, d\theta$

Hence $\int \frac{1}{(a^2+x^2)} dx$

$$= \int \frac{1}{(a^2+a^2\tan^2\theta)}(a\sec^2\theta\, d\theta)$$

$$= \int \frac{a\sec^2\theta\, d\theta}{a^2(1+\tan^2\theta)}$$

$$= \int \frac{a\sec^2\theta\, d\theta}{a^2\sec^2\theta} \text{ since } 1+\tan^2\theta = \sec^2\theta$$

$$= \int \frac{1}{a} d\theta = \frac{1}{a}(\theta) + c$$

Since $x = a\tan\theta$, $\theta = \tan^{-1}\frac{x}{a}$

Hence $\boldsymbol{\int \frac{1}{(a^2+x^2)} dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + c}$

Problem 18. Evaluate: $\int_0^2 \frac{1}{(4+x^2)} dx$

From Problem 17, $\int_0^2 \frac{1}{(4+x^2)} dx$

$$= \frac{1}{2}\left[\tan^{-1}\frac{x}{2}\right]_0^2 \quad \text{since } a = 2$$

$$= \frac{1}{2}(\tan^{-1}1 - \tan^{-1}0) = \frac{1}{2}\left(\frac{\pi}{4} - 0\right)$$

$$= \boldsymbol{\frac{\pi}{8}} \text{ or } \mathbf{0.3927}$$

Problem 19. Evaluate: $\int_0^1 \frac{5}{(3+2x^2)} dx$, correct to 4 decimal places

$$\int_0^1 \frac{5}{(3+2x^2)} dx = \int_0^1 \frac{5}{2[(3/2)+x^2]} dx$$

$$= \frac{5}{2}\int_0^1 \frac{1}{[\sqrt{3/2}]^2+x^2} dx$$

$$= \frac{5}{2}\left[\frac{1}{\sqrt{3/2}}\tan^{-1}\frac{x}{\sqrt{3/2}}\right]_0^1$$

$$= \frac{5}{2}\sqrt{\frac{2}{3}}\left[\tan^{-1}\sqrt{\frac{2}{3}} - \tan^{-1}0\right]$$

$$= (2.0412)[0.6847 - 0]$$

$= \mathbf{1.3976}$, correct to 4 decimal places.

Exercise 8. Integration using the tanθ substitution