Module 8 - Partial Differentiation

A. Introduction to Partial Derivatives

In engineering, it sometimes happens that the variation of one quantity depends on changes taking place in two, or more, other quantities. For example, the volume V of a cylinder is given by $V = \pi r^2 h$. The volume will change if either radius r or height h is changed. The formula for volume may be stated mathematically as $V = f(r, h)$ which means 'V is some function of r and h'. Some other practical examples include:

(i) time of oscillation, $t = 2\pi\sqrt{\dfrac{l}{g}}$ i.e. $t = f(l, g)$.

(ii) torque $T = I\alpha$, i.e. $T = f(I, \alpha)$.

(iii) pressure of an ideal gas $p = \dfrac{mRT}{V}$ i.e. $p = f(T, V)$.

(iv) resonant frequency $f_r = \dfrac{1}{2\pi\sqrt{LC}}$ i.e. $f_r = f(L, C)$, and so on.

When differentiating a function having two variables, one variable is kept constant and the differential coefficient of the other variable is found with respect to that variable. The differential coefficient obtained is called a **partial derivative** of the function.

B. First order partial derivatives

A 'curly dee', ∂, is used to denote a differential coefficient in an expression containing more than one variable.

Hence if $V = \pi r^2 h$ then $\dfrac{\partial V}{\partial r}$ means 'the partial derivative of V with respect to r, with h remaining constant'. Thus,

$$\frac{\partial V}{\partial r} = (\pi h)\frac{\mathrm{d}}{\mathrm{d}r}(r^2) = (\pi h)(2r) = 2\pi rh.$$

Similarly, $\dfrac{\partial V}{\partial h}$ means 'the partial derivative of V with respect to h, with r remaining constant'. Thus,

$$\frac{\partial V}{\partial h} = (\pi r^2)\frac{\mathrm{d}}{\mathrm{d}h}(h) = (\pi r^2)(1) = \pi r^2.$$

$\dfrac{\partial V}{\partial r}$ and $\dfrac{\partial V}{\partial h}$ are examples of **first order partial derivatives**, since $n = 1$ when written in the form $\dfrac{\partial^n V}{\partial r^n}$.

First order partial derivatives are used when finding the total differential, rates of change and errors for functions of two or more variables, when finding maxima, minima and saddle points for functions of two variables, and with partial differential equations.

Problem 1. If $z = 5x^4 + 2x^3y^2 - 3y$ find (a) $\dfrac{\partial z}{\partial x}$ and (b) $\dfrac{\partial z}{\partial y}$

(a) To find $\dfrac{\partial z}{\partial x}$, y is kept constant.

Since $z = 5x^4 + (2y^2)x^3 - (3y)$

then,

$$\frac{\partial z}{\partial x} = \frac{\mathrm{d}}{\mathrm{d}x}(5x^4) + (2y^2)\frac{\mathrm{d}}{\mathrm{d}x}(x^3) - (3y)\frac{\mathrm{d}}{\mathrm{d}x}(1)$$
$$= 20x^3 + (2y^2)(3x^2) - 0.$$

Hence $\boldsymbol{\dfrac{\partial z}{\partial x} = 20x^3 + 6x^2y^2}$.

(b) To find $\dfrac{\partial z}{\partial y}$, x is kept constant.

Since $z = (5x^4) + (2x^3)y^2 - 3y$

then,

$$\frac{\partial z}{\partial y} = (5x^4)\frac{\mathrm{d}}{\mathrm{d}y}(1) + (2x^3)\frac{\mathrm{d}}{\mathrm{d}y}(y^2) - 3\frac{\mathrm{d}}{\mathrm{d}y}(y)$$
$$= 0 + (2x^3)(2y) - 3$$

Hence $\boldsymbol{\frac{\partial z}{\partial y} = 4x^3y - 3}$.

Problem 2. Given $y = 4\sin 3x \cos 2t$, find $\frac{\partial y}{\partial x}$ and $\frac{\partial y}{\partial t}$.

To find $\frac{\partial y}{\partial x}$, t is kept constant

$$\text{Hence} \quad \frac{\partial y}{\partial x} = (4\cos 2t)\frac{d}{dx}(\sin 3x)$$
$$= (4\cos 2t)(3\cos 3x)$$
$$\text{i.e.} \quad \boldsymbol{\frac{\partial y}{\partial x} = 12\cos 3x \cos 2t}$$

To find $\frac{\partial y}{\partial t}$, x is kept constant.

$$\text{Hence} \quad \frac{\partial y}{\partial t} = (4\sin 3x)\frac{d}{dt}(\cos 2t)$$
$$= (4\sin 3x)(-2\sin 2t)$$
$$\text{i.e.} \quad \boldsymbol{\frac{\partial y}{\partial t} = -8\sin 3x \sin 2t}$$

Problem 3. If $z = \sin xy$ show that $\frac{1}{y}\frac{\partial z}{\partial x} = \frac{1}{x}\frac{\partial z}{\partial y}$

$\frac{\partial z}{\partial x} = y\cos xy$, since y is kept constant.

$\frac{\partial z}{\partial y} = x\cos xy$, since x is kept constant.

$$\frac{1}{y}\frac{\partial z}{\partial x} = \left(\frac{1}{y}\right)(y\cos xy) = \cos xy$$

$$\text{and} \quad \frac{1}{x}\frac{\partial z}{\partial y} = \left(\frac{1}{x}\right)(x\cos xy) = \cos xy.$$

$$\textbf{Hence} \quad \boldsymbol{\frac{1}{y}\frac{\partial z}{\partial x} = \frac{1}{x}\frac{\partial z}{\partial y}}$$

Problem 4. Determine $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ when $z = \frac{1}{\sqrt{(x^2+y^2)}}$.

$$z = \frac{1}{\sqrt{(x^2+y^2)}} = (x^2+y^2)^{\frac{-1}{2}}$$

$$\frac{\partial z}{\partial x} = -\frac{1}{2}(x^2+y^2)^{\frac{-3}{2}}(2x), \text{ by the function of a function rule (keeping } y \text{ constant)}$$
$$= \frac{-x}{(x^2+y^2)^{\frac{3}{2}}} = \boldsymbol{\frac{-x}{\sqrt{(x^2+y^2)^3}}}$$

$$\frac{\partial z}{\partial y} = -\frac{1}{2}(x^2+y^2)^{\frac{-3}{2}}(2y), \text{ (keeping } x \text{ constant)}$$
$$= \boldsymbol{\frac{-y}{\sqrt{(x^2+y^2)^3}}}$$

Problem 5. Pressure p of a mass of gas is given by $pV = mRT$, where m and R are constants, V is the volume and T the temperature. Find expressions for $\frac{\partial p}{\partial T}$ and $\frac{\partial p}{\partial V}$

Since $pV = mRT$ then $p = \frac{mRT}{V}$

To find $\frac{\partial p}{\partial T}$, V is kept constant.

$$\text{Hence} \quad \frac{\partial p}{\partial T} = \left(\frac{mR}{V}\right)\frac{d}{dT}(T) = \boldsymbol{\frac{mR}{V}}$$

To find $\frac{\partial p}{\partial V}$, T is kept constant.

$$\text{Hence} \quad \frac{\partial p}{\partial V} = (mRT)\frac{d}{dV}\left(\frac{1}{V}\right)$$
$$= (mRT)(-V^{-2}) = \boldsymbol{\frac{-mRT}{V^2}}$$

Problem 6. The time of oscillation, t, of a pendulum is given by $t = 2\pi\sqrt{\frac{l}{g}}$ where l is the length of the pendulum and g the free fall acceleration due to gravity. Determine $\frac{\partial t}{\partial l}$ and $\frac{\partial t}{\partial g}$

To find $\frac{\partial t}{\partial l}$, g is kept constant.

$$t = 2\pi\sqrt{\frac{l}{g}} = \left(\frac{2\pi}{\sqrt{g}}\right)\sqrt{l} = \left(\frac{2\pi}{\sqrt{g}}\right) l^{\frac{1}{2}}$$

Hence $$\frac{\partial t}{\partial l} = \left(\frac{2\pi}{\sqrt{g}}\right)\frac{\mathrm{d}}{\mathrm{d}l}(l^{\frac{1}{2}}) = \left(\frac{2\pi}{\sqrt{g}}\right)\left(\frac{1}{2}l^{\frac{-1}{2}}\right)$$

$$= \left(\frac{2\pi}{\sqrt{g}}\right)\left(\frac{1}{2\sqrt{l}}\right) = \boldsymbol{\frac{\pi}{\sqrt{lg}}}$$

To find $\frac{\partial t}{\partial g}$, l is kept constant

$$t = 2\pi\sqrt{\frac{l}{g}} = (2\pi\sqrt{l})\left(\frac{1}{\sqrt{g}}\right)$$

$$= (2\pi\sqrt{l})g^{\frac{-1}{2}}$$

Hence $$\frac{\partial t}{\partial g} = (2\pi\sqrt{l})\left(-\frac{1}{2}g^{\frac{-3}{2}}\right)$$

$$= (2\pi\sqrt{l})\left(\frac{-1}{2\sqrt{g^3}}\right)$$

$$= \frac{-\pi\sqrt{l}}{\sqrt{g^3}} = \boldsymbol{-\pi\sqrt{\frac{l}{g^3}}}$$

Exercise 23. First order partial derivatives

C. Second Order Partial Derivatives

As with ordinary differentiation, where a differential coefficient may be differentiated again, a partial derivative may be differentiated partially again to give higher order partial derivatives.

(i) Differentiating $\frac{\partial V}{\partial r}$ of Section 34.2 with respect to r, keeping h constant, gives $\frac{\partial}{\partial r}\left(\frac{\partial V}{\partial r}\right)$ which is written as $\frac{\partial^2 V}{\partial r^2}$

Thus if $V = \pi r^2 h$,

then $\frac{\partial^2 V}{\partial r^2} = \frac{\partial}{\partial r}(2\pi r h) = \mathbf{2\pi h}$.

(ii) Differentiating $\frac{\partial V}{\partial h}$ with respect to h, keeping r constant, gives $\frac{\partial}{\partial h}\left(\frac{\partial V}{\partial h}\right)$ which is written as $\frac{\partial^2 V}{\partial h^2}$

Thus $\frac{\partial^2 V}{\partial h^2} = \frac{\partial}{\partial h}(\pi r^2) = \mathbf{0}$.

(iii) Differentiating $\frac{\partial V}{\partial h}$ with respect to r, keeping h constant, gives $\frac{\partial}{\partial r}\left(\frac{\partial V}{\partial h}\right)$ which is written as $\frac{\partial^2 V}{\partial r \partial h}$. Thus,

$$\frac{\partial^2 V}{\partial r \partial h} = \frac{\partial}{\partial r}\left(\frac{\partial V}{\partial h}\right) = \frac{\partial}{\partial r}(\pi r^2) = \mathbf{2\pi r}.$$

(iv) Differentiating $\frac{\partial V}{\partial r}$ with respect to h, keeping r constant, gives $\frac{\partial}{\partial h}\left(\frac{\partial V}{\partial r}\right)$, which is written as $\frac{\partial^2 V}{\partial h \partial r}$. Thus,

$$\frac{\partial^2 V}{\partial h \partial r} = \frac{\partial}{\partial h}\left(\frac{\partial V}{\partial r}\right) = \frac{\partial}{\partial h}(2\pi r h) = \mathbf{2\pi r}.$$

(v) $\frac{\partial^2 V}{\partial r^2}$, $\frac{\partial^2 V}{\partial h^2}$, $\frac{\partial^2 V}{\partial r \partial h}$ and $\frac{\partial^2 V}{\partial h \partial r}$ are examples of **second order partial derivatives**.

(vi) It is seen from (iii) and (iv) that $\frac{\partial^2 V}{\partial r \partial h} = \frac{\partial^2 V}{\partial h \partial r}$ and such a result is always true for continuous functions (i.e. a graph of the function which has no sudden jumps or breaks).

Second order partial derivatives are used in the solution of partial differential equations, in waveguide theory, in such areas of thermodynamics covering entropy and the continuity theorem, and when finding maxima, minima and saddle points for functions of two variables.

Problem 7. Given $z = 4x^2y^3 - 2x^3 + 7y^2$ find (a) $\frac{\partial^2 z}{\partial x^2}$ (b) $\frac{\partial^2 z}{\partial y^2}$ (c) $\frac{\partial^2 z}{\partial x \partial y}$ (d) $\frac{\partial^2 z}{\partial y \partial x}$

(a)
$$\frac{\partial z}{\partial x} = 8xy^3 - 6x^2$$
$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}(8xy^3 - 6x^2) = \mathbf{8y^3 - 12x}$$

(b)
$$\frac{\partial z}{\partial y} = 12x^2y^2 + 14y$$
$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial y}(12x^2y^2 + 14y) = \mathbf{24x^2y + 14}$$

(c)
$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}(12x^2y^2 + 14y) = \mathbf{24xy^2}$$

(d)
$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial y}(8xy^3 - 6x^2) = \mathbf{24xy^2}$$

$$\left[\text{It is noted that } \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}\right]$$

Problem 8. Show that when $z = e^{-t} \sin\theta$, (a) $\frac{\partial^2 z}{\partial t^2} = -\frac{\partial^2 z}{\partial \theta^2}$, and (b) $\frac{\partial^2 z}{\partial t \partial \theta} = \frac{\partial^2 z}{\partial \theta \partial t}$

(a) $\frac{\partial z}{\partial t} = -e^{-t} \sin\theta$ and $\frac{\partial^2 z}{\partial t^2} = e^{-t} \sin\theta$

$\frac{\partial z}{\partial \theta} = e^{-t} \cos\theta$ and $\frac{\partial^2 z}{\partial \theta^2} = -e^{-t} \sin\theta$

Hence $\boldsymbol{\frac{\partial^2 z}{\partial t^2} = -\frac{\partial^2 z}{\partial \theta^2}}$

(b) $$\frac{\partial^2 z}{\partial t \partial \theta} = \frac{\partial}{\partial t}\left(\frac{\partial z}{\partial \theta}\right) = \frac{\partial}{\partial t}(e^{-t} \cos\theta) = -e^{-t}\cos\theta$$

$$\frac{\partial^2 z}{\partial \theta \partial t} = \frac{\partial}{\partial \theta}\left(\frac{\partial z}{\partial t}\right) = \frac{\partial}{\partial \theta}(-e^{-t} \sin\theta) = -e^{-t}\cos\theta$$

Hence $\boldsymbol{\frac{\partial^2 z}{\partial t \partial \theta} = \frac{\partial^2 z}{\partial \theta \partial t}}$

Problem 9. Show that if $z = \frac{x}{y} \ln y$, then (a) $\frac{\partial z}{\partial y} = x \frac{\partial^2 z}{\partial y \partial x}$ and (b) evaluate $\frac{\partial^2 z}{\partial y^2}$ when $x = -3$ and $y = 1$.

(a) To find $\frac{\partial z}{\partial x}$, y is kept constant.

Hence $\frac{\partial z}{\partial x} = \left(\frac{1}{y} \ln y\right) \frac{d}{dx}(x) = \frac{1}{y} \ln y$

To find $\frac{\partial z}{\partial y}$, x is kept constant.

Hence

$$\frac{\partial z}{\partial y} = (x)\frac{d}{dy}\left(\frac{\ln y}{y}\right)$$

$$= (x)\left\{\frac{(y)\left(\frac{1}{y}\right) - (\ln y)(1)}{y^2}\right\}$$

using the quotient rule

$$= x\left(\frac{1 - \ln y}{y^2}\right) = \frac{x}{y^2}(1 - \ln y)$$

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial y}\left(\frac{\ln y}{y}\right)$$

$$= \frac{(y)\left(\frac{1}{y}\right) - (\ln y)(1)}{y^2}$$

using the quotient rule

$$= \frac{1}{y^2}(1 - \ln y)$$

Hence $\boldsymbol{x \frac{\partial^2 z}{\partial y \partial x} = \frac{x}{y^2}(1 - \ln y) = \frac{\partial z}{\partial y}}$

(b) $$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial y}\left\{\frac{x}{y^2}(1 - \ln y)\right\}$$

$$= (x)\frac{d}{dy}\left(\frac{1 - \ln y}{y^2}\right)$$

$$= (x)\left\{\frac{(y^2)\left(-\frac{1}{y}\right) - (1 - \ln y)(2y)}{y^4}\right\}$$

using the quotient rule

$$= \frac{x}{y^4}[-y - 2y + 2y \ln y]$$

$$= \frac{xy}{y^4}[-3 + 2\ln y] = \frac{x}{y^3}(2\ln y - 3)$$

When $x = -3$ and $y = 1$,

$$\frac{\partial^2 z}{\partial y^2} = \frac{(-3)}{(1)^3}(2\ln 1 - 3) = (-3)(-3) = \mathbf{9}$$

Exercise 24. Second order partial derivatives