Module 8
Integration using Algebraic Substitutions

A. Introduction

Functions that require integrating are not always in the 'standard form'. However, it is often possible to change a function into a form which can be integrated by using either:

(i) an algebraic substitution

(ii) trigonometric substitutions,

(iii) partial fractions,

(iv) the $t = \tan\frac{\theta}{2}$ substitution, or

(v) integration by parts

B. Algebraic substitutions

With **algebraic substitutions**, the substitution usually made is to let u be equal to $f(x)$ such that $f(u)\,du$ is a standard integral. It is found that integrals of the forms:

$$k\int [f(x)]^n f'(x)dx \text{ and } k\int \frac{f'(x)^n}{[f(x)]}dx$$

(where k and n are constants) can both be integrated by substituting u for $f(x)$.

Solved problems on integration using algebraic substitutions

Problem 1. Determine: $\int \cos(3x+7)\,dx$

$\int \cos(3x+7)\,dx$ is not a standard integral of the form shown in Table 1, thus an algebraic substitution is made.

Let $u = 3x+7$ then $\frac{du}{dx} = 3$ and rearranging gives $dx = \frac{du}{3}$

Hence
$$\int \cos(3x+7)\,dx = \int (\cos u)\frac{du}{3} = \int \frac{1}{3}\cos u\,du,$$
which is a standard integral
$$= \frac{1}{3}\sin u + c$$

Rewriting u as $(3x+7)$ gives:
$$\int \cos(3x+7)\,dx = \mathbf{\frac{1}{3}\sin(3x+7)+c},$$
which may be checked by differentiating it.

Problem 2. Find: $\int (2x-5)^7 dx$

$(2x-5)$ may be multiplied by itself 7 times and then each term of the result integrated. However, this would be a lengthy process, and thus an algebraic substitution is made.

Let $u = (2x-5)$ then $\frac{du}{dx} = 2$ and $dx = \frac{du}{2}$

Hence
$$\int (2x-5)^7 dx = \int u^7 \frac{du}{2} = \frac{1}{2}\int u^7 du$$
$$= \frac{1}{2}\left(\frac{u^8}{8}\right) + c = \frac{1}{16}u^8 + c$$

Rewriting u as $(2x-5)$ gives:

$$\int (\mathbf{2x-5})^7 \mathbf{dx} = \frac{\mathbf{1}}{\mathbf{16}}(\mathbf{2x-5})^8 + \boldsymbol{c}$$

Problem 3. Find: $\int \frac{4}{(5x-3)} dx$

Let $u=(5x-3)$ then $\frac{du}{dx}=5$ and $dx=\frac{du}{5}$

Hence $\int \frac{4}{(5x-3)} dx = \int \frac{4}{u}\frac{du}{5} = \frac{4}{5}\int \frac{1}{u} du$

$$= \frac{4}{5}\ln u + c$$

$$= \frac{\mathbf{4}}{\mathbf{5}}\mathbf{\ln(5x-3)} + \boldsymbol{c}$$

Problem 4. Evaluate: $\int_0^1 2e^{6x-1} dx$, correct to 4 significant figures

Let $u=6x-1$ then $\frac{du}{dx}=6$ and $dx=\frac{du}{6}$

Hence $\int 2e^{6x-1} dx = \int 2e^u \frac{du}{6} = \frac{1}{3}\int e^u du$

$$= \frac{1}{3}e^u + c = \frac{1}{3}e^{6x-1} + c$$

Thus $\int_0^1 2e^{6x-1} dx = \frac{1}{3}\left[e^{6x-1}\right]_0^1$

$$= \frac{1}{3}[e^5 - e^{-1}] = \mathbf{49.35},$$

correct to 4 significant figures.

Problem 5. Determine: $\int 3x(4x^2+3)^5 dx$

Let $u=(4x^2+3)$ then $\frac{du}{dx}=8x$ and $dx=\frac{du}{8x}$

Hence

$$\int 3x(4x^2+3)^5 dx = \int 3x(u)^5 \frac{du}{8x}$$

$$= \frac{3}{8}\int u^5 du, \text{ by cancelling}$$

The original variable 'x' has been completely removed and the integral is now only in terms of u and is a standard integral.

Hence $\frac{3}{8}\int u^5 du = \frac{3}{8}\left(\frac{u^6}{6}\right) + c = \frac{1}{16}u^6 + c$

$$= \frac{\mathbf{1}}{\mathbf{16}}(\mathbf{4x^2+3})^6 + \boldsymbol{c}$$

Problem 6. Evaluate: $\int_0^{\pi/6} 24\sin^5\theta\cos\theta d\theta$

Let $u=\sin\theta$ then $\frac{du}{d\theta}=\cos\theta$ and $d\theta=\frac{du}{\cos\theta}$

Hence $\int 24\sin^5\theta\cos\theta\, d\theta$

$$= \int 24u^5 \cos\theta \frac{du}{\cos\theta}$$

$$= 24\int u^5 du, \text{ by cancelling}$$

$$= 24\frac{u^6}{6} + c = 4u^6 + c = 4(\sin\theta)^6 + c$$

$$= 4\sin^6\theta + c$$

Thus $\int_0^{\pi/6} 24\sin^5\theta\cos\theta\, d\theta$

$$= \left[4\sin^6\theta\right]_0^{\pi/6} = 4\left[\left(\sin\frac{\pi}{6}\right)^6 - (\sin 0)^6\right]$$

$$= 4\left[\left(\frac{1}{2}\right)^6 - 0\right] = \frac{\mathbf{1}}{\mathbf{16}} \text{ or } \mathbf{0.0625}$$

Exercise 3. Integration using algebraic substitutions

Solved problems on integration using algebraic substitutions

Problem 7. Find: $\int \frac{x}{2+3x^2}dx$

Let $u = 2+3x^2$ then $\frac{du}{dx} = 6x$ and $dx = \frac{du}{6x}$

Hence $\int \frac{x}{2+3x^2}dx$

$$= \int \frac{x}{u}\frac{du}{6x} = \frac{1}{6}\int \frac{1}{u}du, \text{ by cancelling,}$$

$$= \frac{1}{6}\ln u + x$$

$$= \mathbf{\frac{1}{6}\ln(2+3x^2)+c}$$

Problem 8. Determine: $\int \frac{2x}{\sqrt{4x^2-1}}dx$

Let $u = 4x^2-1$ then $\frac{du}{dx} = 8x$ and $dx = \frac{du}{8x}$

Hence $\int \frac{2x}{\sqrt{4x^2-1}}dx$

$$= \int \frac{2x}{\sqrt{u}}\frac{du}{8x} = \frac{1}{4}\int \frac{1}{\sqrt{u}}du, \text{ by cancelling}$$

$$= \frac{1}{4}\int u^{-1/2}du$$

$$= \frac{1}{4}\left[\frac{u^{(-1/2)+1}}{-\frac{1}{2}+1}\right]+c = \frac{1}{4}\left[\frac{u^{1/2}}{\frac{1}{2}}\right]+c$$

$$= \frac{1}{2}\sqrt{u}+c = \mathbf{\frac{1}{2}\sqrt{4x^2-1}+c}$$

Problem 9. Show that:

$$\int \tan\theta\, d\theta = \ln(\sec\theta)+c$$

$$\int \tan\theta\, d\theta = \int \frac{\sin\theta}{\cos\theta}d\theta$$

Let u = $\cos\theta$

then $\frac{du}{d\theta} = -\sin\theta$ and $d\theta = \frac{-du}{\sin\theta}$

Hence

$$\int \frac{\sin\theta}{\cos\theta}d\theta = \int \frac{\sin\theta}{u}\left(\frac{-du}{\sin\theta}\right)$$

$$= -\int \frac{1}{u}du = -\ln u + c$$

$$= -\ln(\cos\theta)+c$$

$$= \ln(\cos\theta)^{-1}+c,$$

by the laws of logarithms

Hence $\int \boldsymbol{\tan\theta\, d\theta = \ln(\sec\theta)+c}$,

since $(\cos\theta)^{-1} = \frac{1}{\cos\theta} = \sec\theta$

C. Change of limits

When evaluating definite integrals involving substitutions it is sometimes more convenient to **change the limits** of the integral as shown in Problems 10 and 11.

Problem 10. Evaluate: $\int_1^3 5x\sqrt{2x^2+7}\,dx$, taking positive values of square roots only

Let $u=2x^2+7$, then $\frac{du}{dx}=4x$ and $dx=\frac{du}{4x}$

It is possible in this case to change the limits of integration. Thus when $x=3$, $u=2(3)^2+7=25$ and when $x=1$, $u=2(1)^2+7=9$

$$\text{Hence} \quad \int_{x=1}^{x=3} 5x\sqrt{2x^2+7}\,dx$$

$$= \int_{u=9}^{u=25} 5x\sqrt{u}\,\frac{du}{4x} = \frac{5}{4}\int_9^{25}\sqrt{u}\,du$$

$$= \frac{5}{4}\int_9^{25} u^{1/2}du$$

Thus the limits have been changed, and it is unnecessary to change the integral back in terms of x.

$$\text{Thus} \quad \int_{x=1}^{x=3} 5x\sqrt{2x^2+7}\,dx$$

$$= \frac{5}{4}\left[\frac{u^{3/2}}{3/2}\right]_9^{25} = \frac{5}{6}\left[\sqrt{u^3}\right]_9^{25}$$

$$= \frac{5}{6}[\sqrt{25^3}-\sqrt{9^3}] = \frac{5}{6}(125-27) = \mathbf{81\frac{2}{3}}$$

Problem 11. Evaluate: $\int_0^2 \frac{3x}{\sqrt{2x^2+1}}\,dx$, taking positive values of square roots only

Let $u=2x^2+1$ then $\frac{du}{dx}=4x$ and $dx=\frac{du}{4x}$

$$\text{Hence} \quad \int_0^2 \frac{3x}{\sqrt{2x^2+1}}\,dx = \int_{x=0}^{x=2}\frac{3x}{\sqrt{u}}\,\frac{du}{4x}$$

$$= \frac{3}{4}\int_{x=0}^{x=2} u^{-1/2}du$$

Since $u=2x^2+1$, when $x=2$, $u=9$ and when $x=0$, $u=1$

Thus $\frac{3}{4}\int_{x=0}^{x=2} u^{-1/2}\,du = \frac{3}{4}\int_{u=1}^{u=9} u^{-1/2}\,du$,

i.e. the limits have been changed

$$= \frac{3}{4}\left[\frac{u^{1/2}}{\frac{1}{2}}\right]_1^9 = \frac{3}{2}[\sqrt{9}-\sqrt{1}] = \mathbf{3},$$

taking positive values of square roots only.

Exercise 4. Integration using algebraic substitutions