Module 7 - Inverse Trigonometric & Hyperbolic Functions

A. Inverse Functions

If $y = 3x - 2$, then by transposition, $x = \frac{y+2}{3}$. The function $x = \frac{y+2}{3}$ is called the **inverse function** of $y = 3x - 2$

Inverse trigonometric functions are denoted by prefixing the function with 'arc' or, more commonly, by using the $^{-1}$ notation. For example, if $y = \sin x$, then $x = \arcsin y$ or $x = \sin^{-1} y$. Similarly, if $y = \cos x$, then $x = \arccos y$ or $x = \cos^{-1} y$, and so on. In this chapter the $^{-1}$ notation will be used. A sketch of each of the inverse trigonometric functions is shown in Fig. 21.

Inverse hyperbolic functions are denoted by prefixing the function with 'ar' or, more commonly, by using the $^{-1}$ notation. For example, if $y = \sinh x$, then $x = \text{arsinh}\, y$ or $x = \sinh^{-1} y$. Similarly, if $y = \text{sech}\, x$, then $x = \text{arsech}\, y$ or $x = \text{sech}^{-1} y$, and so on. In this chapter the $^{-1}$ notation will be used. A sketch of each of the inverse hyperbolic functions is shown in Fig. 22.

B. Differentiation of Inverse Trigonometric Functions

(i) If $y = \sin^{-1} x$, then $x = \sin y$.
Differentiating both sides with respect to y gives:

$$\frac{dx}{dy} = \cos y = \sqrt{1 - \sin^2 y}$$

since $\cos^2 y + \sin^2 y = 1$, i.e. $\frac{dx}{dy} = \sqrt{1 - x^2}$

However $\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$

Hence, when $y = \sin^{-1} x$ then

$$\mathbf{\frac{dy}{dx} = \frac{1}{\sqrt{1 - x^2}}}$$

(ii) A sketch of part of the curve of $y = \sin^{-1} x$ is shown in Fig. 21(a). The principal value of $\sin^{-1} x$ is defined as the value lying between $-\pi/2$ and $\pi/2$. The gradient of the curve between points A and B is positive for all values of x and thus only the positive value is taken

when evaluating $\frac{1}{\sqrt{1 - x^2}}$.

(iii) Given $y = \sin^{-1} \frac{x}{a}$ then $\frac{x}{a} = \sin y$ and $x = a \sin y$

Hence $\frac{dx}{dy} = a \cos y = a\sqrt{1 - \sin^2 y}$

$$= a\sqrt{\left[1 - \left(\frac{x}{a}\right)^2\right]} = a\sqrt{\left(\frac{a^2 - x^2}{a^2}\right)}$$

$$= \frac{a\sqrt{a^2 - x^2}}{a} = \sqrt{a^2 - x^2}$$

Thus $\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{\sqrt{a^2 - x^2}}$

i.e. **when $y = \sin^{-1} \frac{x}{a}$ then $\frac{dy}{dx} = \frac{1}{\sqrt{a^2 - x^2}}$**

Since integration is the reverse process of differentiation then:

$$\int \frac{1}{\sqrt{a^2 - x^2}}\, dx = \sin^{-1} \frac{x}{a} + c$$

(iv) Given $y = \sin^{-1} f(x)$ the function of a function rule may be used to find $\frac{dy}{dx}$.

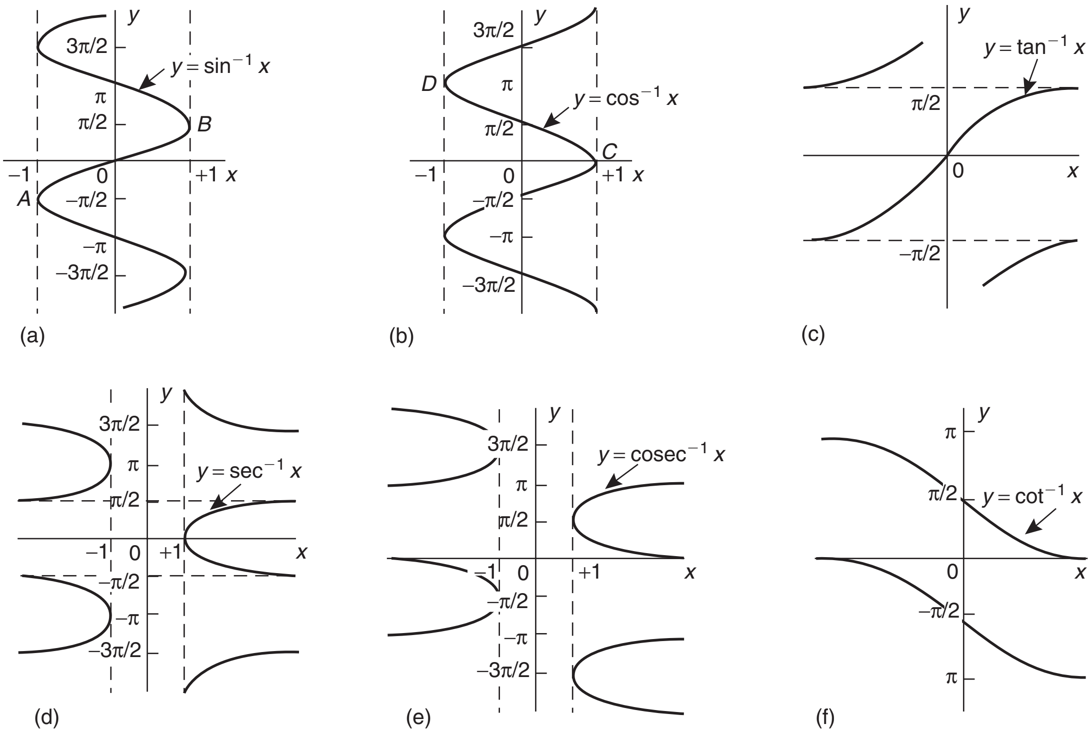

Figure 21

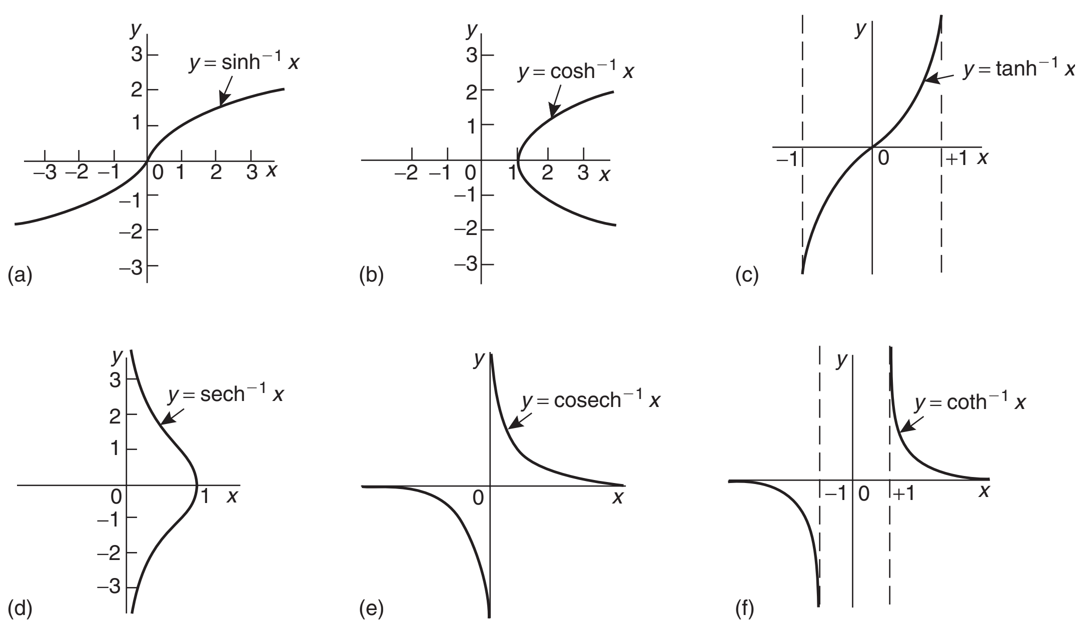

Figure 22

Let $u = f(x)$ then $y = \sin^{-1} u$

Then $\frac{du}{dx} = f'(x)$ and $\frac{dy}{du} = \frac{1}{\sqrt{1-u^2}}$

(see para. (i))

Thus $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \frac{1}{\sqrt{1-u^2}} f'(x)$

$$= \boldsymbol{\frac{f'(x)}{\sqrt{1-[f(x)]^2}}}$$

(v) The differential coefficients of the remaining inverse trigonometric functions are obtained in a similar manner to that shown above and a summary of the results is shown in Table 1.

Table 1. Differential coefficients of inverse trigonometric functions

y or $f(x)$	$\frac{dy}{dx}$ or $f'(x)$
(i) $\sin^{-1} \frac{x}{a}$	$\frac{1}{\sqrt{a^2-x^2}}$
$\sin^{-1} f(x)$	$\frac{f'(x)}{\sqrt{1-[f(x)]^2}}$
(ii) $\cos^{-1} \frac{x}{a}$	$\frac{-1}{\sqrt{a^2-x^2}}$
$\cos^{-1} f(x)$	$\frac{-f'(x)}{\sqrt{1-[f(x)]^2}}$
(iii) $\tan^{-1} \frac{x}{a}$	$\frac{a}{a^2+x^2}$
$\tan^{-1} f(x)$	$\frac{f'(x)}{1+[f(x)]^2}$
(iv) $\sec^{-1} \frac{x}{a}$	$\frac{a}{x\sqrt{x^2-a^2}}$
$\sec^{-1} f(x)$	$\frac{f'(x)}{f(x)\sqrt{[f(x)]^2-1}}$
(v) $\operatorname{cosec}^{-1} \frac{x}{a}$	$\frac{-a}{x\sqrt{x^2-a^2}}$
$\operatorname{cosec}^{-1} f(x)$	$\frac{-f'(x)}{f(x)\sqrt{[f(x)]^2-1}}$
(vi) $\cot^{-1} \frac{x}{a}$	$\frac{-a}{a^2+x^2}$
$\cot^{-1} f(x)$	$\frac{-f'(x)}{1+[f(x)]^2}$

Problem 1. Find $\frac{dy}{dx}$ given $y = \sin^{-1} 5x^2$.

From Table 1(i), if

$$y = \sin^{-1} f(x) \text{ then } \frac{dy}{dx} = \frac{f'(x)}{\sqrt{1-[f(x)]^2}}$$

Hence, if $y = \sin^{-1} 5x^2$ then $f(x) = 5x^2$ and $f'(x) = 10x$.

Thus $\frac{dy}{dx} = \frac{10x}{\sqrt{1-(5x^2)^2}} = \boldsymbol{\frac{10x}{\sqrt{1-25x^4}}}$

Problem 2.

(a) Show that if $y = \cos^{-1} x$ then $\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$

(b) Hence obtain the differential coefficient of $y = \cos^{-1}(1-2x^2)$.

(a) If $y = \cos^{-1} x$ then $x = \cos y$.

Differentiating with respect to y gives:

$$\frac{dx}{dy} = -\sin y = -\sqrt{1-\cos^2 y}$$
$$= -\sqrt{1-x^2}$$

Hence $\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \boldsymbol{-\frac{1}{\sqrt{1-x^2}}}$

The principal value of $y = \cos^{-1} x$ is defined as the angle lying between 0 and π, i.e. between points C and D shown in Fig. 21(b). The gradient of the curve is negative between C and D and thus the differential coefficient $\frac{dy}{dx}$ is negative as shown above.

(b) If $y = \cos^{-1} f(x)$ then by letting $u = f(x)$, $y = \cos^{-1} u$

Then $\frac{dy}{du} = -\frac{1}{\sqrt{1-u^2}}$ (from part (a))

and $\frac{du}{dx} = f'(x)$

From the function of a function rule,

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = -\frac{1}{\sqrt{1-u^2}} f'(x)$$
$$= \frac{-f'(x)}{\sqrt{1-[f(x)]^2}}$$

Hence, when $y = \cos^{-1}(1 - 2x^2)$

then $\dfrac{dy}{dx} = \dfrac{-(-4x)}{\sqrt{1 - [1 - 2x^2]^2}}$

$$= \frac{4x}{\sqrt{1 - (1 - 4x^2 + 4x^4)}} = \frac{4x}{\sqrt{(4x^2 - 4x^4)}}$$

$$= \frac{4x}{\sqrt{[4x^2(1 - x^2)]}} = \frac{4x}{2x\sqrt{1 - x^2}} = \mathbf{\frac{2}{\sqrt{1 - x^2}}}$$

Problem 3. Determine the differential coefficient of $y = \tan^{-1}\dfrac{x}{a}$ and show that the differential coefficient of $\tan^{-1}\dfrac{2x}{3}$ is $\dfrac{6}{9 + 4x^2}$

If $y = \tan^{-1}\dfrac{x}{a}$ then $\dfrac{x}{a} = \tan y$ and $x = a\tan y$

$\dfrac{dx}{dy} = a\sec^2 y = a(1 + \tan^2 y)$ since

$$\sec^2 y = 1 + \tan^2 y$$

$$= a\left[1 + \left(\frac{x}{a}\right)^2\right] = a\left(\frac{a^2 + x^2}{a^2}\right)$$

$$= \frac{a^2 + x^2}{a}$$

Hence $\dfrac{dy}{dx} = \dfrac{1}{\dfrac{dx}{dy}} = \dfrac{a}{a^2 + x^2}$

The principal value of $y = \tan^{-1} x$ is defined as the angle lying between $-\dfrac{\pi}{2}$ and $\dfrac{\pi}{2}$ and the gradient $\left(\text{i.e. } \dfrac{dy}{dx}\right)$ between these two values is always positive (see Fig. 21 (c)).

Comparing $\tan^{-1}\dfrac{2x}{3}$ with $\tan^{-1}\dfrac{x}{a}$ shows that $a = \dfrac{3}{2}$

Hence if $y = \tan^{-1}\dfrac{2x}{3}$ then

$$\frac{dy}{dx} = \frac{\frac{3}{2}}{\left(\frac{3}{2}\right)^2 + x^2} = \frac{\frac{3}{2}}{\frac{9}{4} + x^2} = \frac{\frac{3}{2}}{\frac{9 + 4x^2}{4}}$$

$$= \frac{\frac{3}{2}(4)}{9 + 4x^2} = \mathbf{\frac{6}{9 + 4x^2}}$$

Problem 4. Find the differential coefficient of $y = \ln(\cos^{-1} 3x)$.

Let $u = \cos^{-1} 3x$ then $y = \ln u$.

By the function of a function rule,

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = \frac{1}{u} \times \frac{d}{dx}(\cos^{-1} 3x)$$

$$= \frac{1}{\cos^{-1} 3x}\left\{\frac{-3}{\sqrt{1 - (3x)^2}}\right\}$$

i.e. $$\mathbf{\frac{d}{dx}[\ln(\cos^{-1} 3x)] = \frac{-3}{\sqrt{1 - 9x^2}\,\cos^{-1} 3x}}$$

Problem 5. If $y = \tan^{-1}\dfrac{3}{t^2}$ find $\dfrac{dy}{dt}$

Using the general form from Table 1(iii),

$$f(t) = \frac{3}{t^2} = 3t^{-2},$$

from which $f'(t) = \dfrac{-6}{t^3}$

Hence $\dfrac{d}{dt}\left(\tan^{-1}\dfrac{3}{t^2}\right) = \dfrac{f'(t)}{1 + [f(t)]^2}$

$$= \frac{-\frac{6}{t^3}}{\left\{1 + \left(\frac{3}{t^2}\right)^2\right\}} = \frac{-\frac{6}{t^3}}{\frac{t^4 + 9}{t^4}}$$

$$= \left(-\frac{6}{t^3}\right)\left(\frac{t^4}{t^4 + 9}\right) = \mathbf{-\frac{6t}{t^4 + 9}}$$

Problem 6. Differentiate $y = \dfrac{\cot^{-1} 2x}{1 + 4x^2}$

Using the quotient rule:

$$\frac{dy}{dx} = \frac{(1 + 4x^2)\left(\frac{-2}{1 + (2x)^2}\right) - (\cot^{-1} 2x)(8x)}{(1 + 4x^2)^2}$$

from Table 1(vi)

$$= \mathbf{\frac{-2(1 + 4x\cot^{-1} 2x)}{(1 + 4x^2)^2}}$$

Problem 7. Differentiate $y = x \operatorname{cosec}^{-1} x$.

Using the product rule:

$$\frac{dy}{dx} = (x)\left[\frac{-1}{x\sqrt{x^2-1}}\right] + (\operatorname{cosec}^{-1} x)(1)$$

from Table 1(v)

$$= \mathbf{\frac{-1}{\sqrt{x^2-1}} + \operatorname{cosec}^{-1} x}$$

Problem 8. Show that if

$$y = \tan^{-1}\left(\frac{\sin t}{\cos t - 1}\right) \text{ then } \frac{dy}{dt} = \frac{1}{2}$$

If $\quad f(t) = \left(\dfrac{\sin t}{\cos t - 1}\right)$

then $\quad f'(t) = \dfrac{(\cos t - 1)(\cos t) - (\sin t)(-\sin t)}{(\cos t - 1)^2}$

$$= \frac{\cos^2 t - \cos t + \sin^2 t}{(\cos t - 1)^2} = \frac{1 - \cos t}{(\cos t - 1)^2}$$

since $\sin^2 t + \cos^2 t = 1$

$$= \frac{-(\cos t - 1)}{(\cos t - 1)^2} = \frac{-1}{\cos t - 1}$$

Using Table 1(iii), when

$$y = \tan^{-1}\left(\frac{\sin t}{\cos t - 1}\right)$$

then $\dfrac{dy}{dt} = \dfrac{\dfrac{-1}{\cos t - 1}}{1 + \left(\dfrac{\sin t}{\cos t - 1}\right)^2}$

$$= \frac{\dfrac{-1}{\cos t - 1}}{\dfrac{(\cos t - 1)^2 + (\sin t)^2}{(\cos t - 1)^2}}$$

$$= \left(\frac{-1}{\cos t - 1}\right)\left(\frac{(\cos t - 1)^2}{\cos^2 t - 2\cos t + 1 + \sin^2 t}\right)$$

$$= \frac{-(\cos t - 1)}{2 - 2\cos t} = \frac{1 - \cos t}{2(1 - \cos t)} = \mathbf{\frac{1}{2}}$$

Exercise 20. Differentiating inverse trigonometric functions

C. Logarithmic forms of the inverse hyperbolic functions

Inverse hyperbolic functions may be evaluated most conveniently when expressed in a **logarithmic form**.

For example, if $y = \sinh^{-1}\frac{x}{a}$ then $\frac{x}{a} = \sinh y$.

$e^{y} = \cosh y + \sinh y$ and $\cosh^2 y - \sinh^2 y \equiv 1$, from which,

$\cosh y = \sqrt{1 + \sinh^2 y}$ which is positive since cosh y is always positive.

$$\text{Hence}\quad e^{y} = \sqrt{1+\sinh^2 y} + \sinh y$$

$$= \sqrt{\left[1 + \left(\frac{x}{a}\right)^2\right]} + \frac{x}{a} = \sqrt{\left(\frac{a^2+x^2}{a^2}\right)} + \frac{x}{a}$$

$$= \frac{\sqrt{a^2+x^2}}{a} + \frac{x}{a} \quad \text{or} \quad \frac{x+\sqrt{a^2+x^2}}{a}$$

Taking Napierian logarithms of both sides gives:

$$y = \ln\left\{\frac{x+\sqrt{a^2+x^2}}{a}\right\}$$

$$\text{Hence,}\quad \boldsymbol{\sinh^{-1}\frac{x}{a} = \ln\left\{\frac{x+\sqrt{a^2+x^2}}{a}\right\}} \qquad (1)$$

Thus to evaluate $\sinh^{-1}\frac{3}{4}$, let $x = 3$ and $a = 4$ in equation (1).

$$\text{Then}\ \sinh^{-1}\frac{3}{4} = \ln\left\{\frac{3+\sqrt{4^2+3^2}}{4}\right\}$$

$$= \ln\left(\frac{3+5}{4}\right) = \ln 2 = 0.6931$$

By similar reasoning to the above it may be shown that:

$$\boldsymbol{\cosh^{-1}\frac{x}{a} = \ln\left\{\frac{x+\sqrt{x^2-a^2}}{a}\right\}}$$

$$\text{and}\quad \boldsymbol{\tanh^{-1}\frac{x}{a} = \frac{1}{2}\ln\left(\frac{a+x}{a-x}\right)}$$

Problem 9. Evaluate, correct to 4 decimal places, $\sinh^{-1} 2$.

From above, $\sinh^{-1}\frac{x}{a} = \ln\left\{\frac{x+\sqrt{a^2+x^2}}{a}\right\}$

With $x = 2$ and $a = 1$,

$$\sinh^{-1} 2 = \ln\left\{\frac{2+\sqrt{1^2+2^2}}{1}\right\}$$

$$= \ln(2+\sqrt{5}) = \ln 4.2361$$

$$= \mathbf{1.4436}, \textbf{ correct to 4 decimal places}$$

Problem 10. Show that $\tanh^{-1}\frac{x}{a} = \frac{1}{2}\ln\left(\frac{a+x}{a-x}\right)$ and evaluate, correct to 4 decimal places, $\tanh^{-1}\frac{3}{5}$

If $y = \tanh^{-1}\frac{x}{a}$ then $\frac{x}{a} = \tanh\ y$.

$$\tanh y = \frac{\sinh x}{\cosh x} = \frac{\frac{1}{2}(e^{y}-e^{-y})}{\frac{1}{2}(e^{y}+e^{-y})} = \frac{e^{2y}-1}{e^{2y}+1}$$

by dividing each term by e^{-y}

Thus, $$\frac{x}{a}=\frac{e^{2y}-1}{e^{2y}+1}$$

from which, $x(e^{2y}+1)=a(e^{2y}-1)$

Hence $x+a=ae^{2y}-xe^{2y}=e^{2y}(a-x)$

from which $e^{2y}=\left(\frac{a+x}{a-x}\right)$

Taking Napierian logarithms of both sides gives:

$$2y=\ln\left(\frac{a+x}{a-x}\right)$$

and $$y=\frac{1}{2}\ln\left(\frac{a+x}{a-x}\right)$$

Hence, $\boldsymbol{\tanh^{-1}\frac{x}{a}=\frac{1}{2}\ln\left(\frac{a+x}{a-x}\right)}$

Substituting $x=3$ and $a=5$ gives:

$$\tanh^{-1}\frac{3}{5}=\frac{1}{2}\ln\left(\frac{5+3}{5-3}\right)=\frac{1}{2}\ln 4$$
$$=\mathbf{0.6931}\text{, correct to 4 decimal places}$$

Problem 11. Prove that

$$\cosh^{-1}\frac{x}{a}=\ln\left\{\frac{x+\sqrt{x^2-a^2}}{a}\right\}$$

and hence evaluate $\cosh^{-1}1.4$ correct to 4 decimal places.

If $y=\cosh^{-1}\frac{x}{a}$ then $\frac{x}{a}=\cos y$

$$e^y=\cosh y+\sinh y=\cosh y\pm\sqrt{\cosh^2 y-1}$$
$$=\frac{x}{a}\pm\sqrt{\left[\left(\frac{x}{a}\right)^2-1\right]}=\frac{x}{a}\pm\frac{\sqrt{x^2-a^2}}{a}$$
$$=\frac{x\pm\sqrt{x^2-a^2}}{a}$$

Taking Napierian logarithms of both sides gives:

$$y=\ln\left\{\frac{x\pm\sqrt{x^2-a^2}}{a}\right\}$$

Thus, assuming the principal value,

$$\boldsymbol{\cosh^{-1}\frac{x}{a}=\ln\left\{\frac{x+\sqrt{x^2-a^2}}{a}\right\}}$$

$$\cosh^{-1}1.4=\cosh^{-1}\frac{14}{10}=\cosh^{-1}\frac{7}{5}$$

In the equation for $\cosh^{-1}\frac{x}{a}$, let $x=7$ and $a=5$

$$\text{Then }\cosh^{-1}\frac{7}{5}=\ln\left\{\frac{7+\sqrt{7^2-5^2}}{5}\right\}$$
$$=\ln 2.3798=\mathbf{0.8670},$$
correct to 4 decimal places

Exercise 21. Logarithmic forms of the inverse hyperbolic functions

D. Differentiation of Inverse Hyperbolic Functions

If $y=\sinh^{-1}\frac{x}{a}$ then $\frac{x}{a}=\sinh y$ and $x=a\sinh y$

$$\frac{dx}{dy}=a\cosh y$$

Also $\cosh^2 y-\sinh^2 y=1$, from which,

$$\cosh y=\sqrt{1+\sinh^2 y}=\sqrt{\left[1+\left(\frac{x}{a}\right)^2\right]}$$
$$=\frac{\sqrt{a^2+x^2}}{a}$$

Hence $$\frac{dx}{dy}=a\cosh y=\frac{a\sqrt{a^2+x^2}}{a}=\sqrt{a^2+x^2}$$

Then $$\boldsymbol{\frac{dy}{dx}}=\frac{1}{\frac{dx}{dy}}=\boldsymbol{\frac{1}{\sqrt{a^2+x^2}}}$$

[An alternative method of differentiating $\sinh^{-1}\dfrac{x}{a}$ is to differentiate the logarithmic form $\ln\left\{\dfrac{x+\sqrt{a^2+x^2}}{a}\right\}$ with respect to x].

From the sketch of $y=\sinh^{-1}x$ shown in Fig. 22(a) it is seen that the gradient $\left(\text{i.e. } \dfrac{dy}{dx}\right)$ is always positive.

It follows from above that

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \sinh^{-1}\frac{x}{a}+c$$

or $$\ln\left\{\frac{x+\sqrt{a^2+x^2}}{a}\right\}+c$$

It may be shown that

$$\mathbf{\frac{d}{dx}(\sinh^{-1}x)=\frac{1}{\sqrt{x^2+1}}}$$

or more generally

$$\mathbf{\frac{d}{dx}[\sinh^{-1}f(x)]=\frac{f'(x)}{\sqrt{[f(x)]^2+1}}}$$

by using the function of a function rule.

The remaining inverse hyperbolic functions are differentiated in a similar manner to that shown above and the results are summarized in Table 2.

Table 2 Differential coefficients of inverse hyperbolic functions

y or $f(x)$	$\dfrac{dy}{dx}$ or $f'(x)$
(i) $\sinh^{-1}\dfrac{x}{a}$	$\dfrac{1}{\sqrt{x^2+a^2}}$
$\sinh^{-1}f(x)$	$\dfrac{f'(x)}{\sqrt{[f(x)]^2+1}}$
(ii) $\cosh^{-1}\dfrac{x}{a}$	$\dfrac{1}{\sqrt{x^2-a^2}}$
$\cosh^{-1}f(x)$	$\dfrac{f'(x)}{\sqrt{[f(x)]^2-1}}$
(iii) $\tanh^{-1}\dfrac{x}{a}$	$\dfrac{a}{a^2-x^2}$
$\tanh^{-1}f(x)$	$\dfrac{f'(x)}{1-[f(x)]^2}$
(iv) $\text{sech}^{-1}\dfrac{x}{a}$	$\dfrac{-a}{x\sqrt{a^2-x^2}}$
$\text{sech}^{-1}f(x)$	$\dfrac{-f'(x)}{f(x)\sqrt{1-[f(x)]^2}}$
(v) $\text{cosech}^{-1}\dfrac{x}{a}$	$\dfrac{-a}{x\sqrt{x^2+a^2}}$
$\text{cosech}^{-1}f(x)$	$\dfrac{-f'(x)}{f(x)\sqrt{[f(x)]^2+1}}$
(vi) $\coth^{-1}\dfrac{x}{a}$	$\dfrac{a}{a^2-x^2}$
$\coth^{-1}f(x)$	$\dfrac{f'(x)}{1-[f(x)]^2}$

Problem 12. Find the differential coefficient of $y=\sinh^{-1}2x$.

From Table 2(i),

$$\frac{d}{dx}[\sinh^{-1}f(x)]=\frac{f'(x)}{\sqrt{[f(x)]^2+1}}$$

Hence $$\frac{d}{dx}(\sinh^{-1}2x)=\frac{2}{\sqrt{[(2x)^2+1]}}=\mathbf{\frac{2}{\sqrt{[4x^2+1]}}}$$

Problem 13. Determine $\dfrac{d}{dx}\left[\cosh^{-1}\sqrt{(x^2+1)}\right]$

If $y=\cosh^{-1}f(x)$, $\dfrac{dy}{dx}=\dfrac{f'(x)}{\sqrt{[f(x)]^2-1}}$

If $y=\cosh^{-1}\sqrt{(x^2+1)}$, then $f(x)=\sqrt{(x^2+1)}$ and $f'(x)=\dfrac{1}{2}(x+1)^{-1/2}(2x)=\dfrac{x}{\sqrt{(x^2+1)}}$

Hence, $\dfrac{d}{dx}\left[\cosh^{-1}\sqrt{(x^2+1)}\right]$

$$=\frac{\dfrac{x}{\sqrt{(x^2+1)}}}{\sqrt{\left[\left(\sqrt{(x^2+1)}\right)^2-1\right]}}=\frac{\dfrac{x}{\sqrt{(x^2+1)}}}{\sqrt{(x^2+1-1)}}$$

$$=\frac{\dfrac{x}{\sqrt{(x^2+1)}}}{x}=\mathbf{\frac{1}{\sqrt{(x^2+1)}}}$$

Problem 14. Show that $\frac{d}{dx}\left[\tanh^{-1}\frac{x}{a}\right] = \frac{a}{a^2-x^2}$ and hence determine the differential coefficient of $\tanh^{-1}\frac{4x}{3}$

If $y = \tanh^{-1}\frac{x}{a}$ then $\frac{x}{a} = \tanh y$ and $x = a\tanh y$

$\frac{dx}{dy} = a\,\text{sech}^2 y = a(1-\tanh^2 y)$, since $1-\text{sech}^2 y = \tanh^2 y$

$$= a\left[1-\left(\frac{x}{a}\right)^2\right] = a\left(\frac{a^2-x^2}{a^2}\right) = \frac{a^2-x^2}{a}$$

Hence $\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{a}{a^2-x^2}$

Comparing $\tanh^{-1}\frac{4x}{3}$ with $\tanh^{-1}\frac{x}{a}$ shows that $a = \frac{3}{4}$

Hence $\frac{d}{dx}\left[\tanh^{-1}\frac{4x}{3}\right] = \frac{\frac{3}{4}}{\left(\frac{3}{4}\right)^2 - x^2} = \frac{\frac{3}{4}}{\frac{9}{16}-x^2}$

$$= \frac{\frac{3}{4}}{\frac{9-16x^2}{16}} = \frac{3}{4}\cdot\frac{16}{(9-16x^2)} = \mathbf{\frac{12}{9-16x^2}}$$

Problem 15. Differentiate $\text{cosech}^{-1}(\sinh\theta)$.

From Table 2(v),

$$\frac{d}{dx}\left[\text{cosech}^{-1} f(x)\right] = \frac{-f'(x)}{f(x)\sqrt{[f(x)]^2+1}}$$

Hence $\frac{d}{d\theta}[\text{cosech}^{-1}(\sinh\theta)]$

$$= \frac{-\cosh\theta}{\sinh\theta\sqrt{[\sinh^2\theta+1]}}$$

$$= \frac{-\cosh\theta}{\sinh\theta\sqrt{\cosh^2\theta}} \text{ since } \cosh^2\theta - \sinh^2\theta = 1$$

$$= \frac{-\cosh\theta}{\sinh\theta\cosh\theta} = \frac{-1}{\sinh\theta} = \mathbf{-cosech\,\theta}$$

Problem 16. Find the differential coefficient of $y = \text{sech}^{-1}(2x-1)$.

From Table 2(iv),

$$\frac{d}{dx}[\text{sech}^{-1} f(x)] = \frac{-f'(x)}{f(x)\sqrt{1-[f(x)]^2}}$$

Hence, $\frac{d}{dx}[\text{sech}^{-1}(2x-1)]$

$$= \frac{-2}{(2x-1)\sqrt{[1-(2x-1)^2]}}$$

$$= \frac{-2}{(2x-1)\sqrt{[1-(4x^2-4x+1)]}}$$

$$= \frac{-2}{(2x-1)\sqrt{(4x-4x^2)}} = \frac{-2}{(2x-1)\sqrt{[4x(1-x)]}}$$

$$= \frac{-2}{(2x-1)2\sqrt{[x(1-x)]}} = \mathbf{\frac{-1}{(2x-1)\sqrt{[x(1-x)]}}}$$

Problem 17. Show that $\frac{d}{dx}[\coth^{-1}(\sin x)] = \sec x$.

From Table 2(vi),

$$\frac{d}{dx}[\coth^{-1} f(x)] = \frac{f'(x)}{1-[f(x)]^2}$$

Hence $\frac{d}{dx}[\coth^{-1}(\sin x)] = \frac{\cos x}{[1-(\sin x)^2]}$

$$= \frac{\cos x}{\cos^2 x} \text{ since } \cos^2 x + \sin^2 x = 1$$

$$= \frac{1}{\cos x} = \mathbf{\sec x}$$

Problem 18. Differentiate $y = (x^2-1)\tanh^{-1} x$.

Using the product rule,

$$\frac{dy}{dx} = (x^2-1)\left(\frac{1}{1-x^2}\right) + (\tanh^{-1} x)(2x)$$

$$= \frac{-(1-x^2)}{(1-x^2)} + 2x\tanh^{-1} x = \mathbf{2x\tanh^{-1} x - 1}$$

Problem 19. Determine $\int \frac{dx}{\sqrt{(x^2+4)}}$

Since $\frac{d}{dx}\left(\sinh^{-1}\frac{x}{a}\right) = \frac{1}{\sqrt{(x^2+a^2)}}$

then $\int \frac{dx}{\sqrt{(x^2+a^2)}} = \sinh^{-1}\frac{x}{a} + c$

Hence $\int \frac{1}{\sqrt{(x^2+4)}}\,dx = \int \frac{1}{\sqrt{(x^2+2^2)}}\,dx$

$$= \mathbf{\sinh^{-1}\frac{x}{2} + c}$$

Problem 20. Determine $\int \frac{4}{\sqrt{(x^2-3)}}\,dx$.

Since $\frac{d}{dx}\left(\cosh^{-1}\frac{x}{a}\right) = \frac{1}{\sqrt{(x^2-a^2)}}$

then $\int \frac{1}{\sqrt{(x^2-a^2)}}\,dx = \cosh^{-1}\frac{x}{a} + c$

Hence $\int \frac{4}{\sqrt{(x^2-3)}}\,dx = 4\int \frac{1}{\sqrt{[x^2-(\sqrt{3})^2]}}\,dx$

$$= \mathbf{4\cosh^{-1}\frac{x}{\sqrt{3}} + c}$$

Problem 21. Find $\int \frac{2}{(9-4x^2)}\,dx$.

Since $\tanh^{-1}\frac{x}{a} = \frac{a}{a^2-x^2}$

then $\int \frac{a}{a^2-x^2}\,dx = \tanh^{-1}\frac{x}{a} + c$

i.e. $\int \frac{1}{a^2-x^2}\,dx = \frac{1}{a}\tanh^{-1}\frac{x}{a} + c$

Hence $\int \frac{2}{(9-4x^2)}\,dx = 2\int \frac{1}{4\left(\frac{9}{4}-x^2\right)}\,dx$

$$= \frac{1}{2}\int \frac{1}{\left[\left(\frac{3}{2}\right)^2 - x^2\right]}\,dx$$

$$= \frac{1}{2}\left[\frac{1}{\left(\frac{3}{2}\right)}\tanh^{-1}\frac{x}{\left(\frac{3}{2}\right)} + c\right]$$

i.e. $\int \frac{2}{(9-4x^2)}\,dx = \mathbf{\frac{1}{3}\tanh^{-1}\frac{2x}{3} + c}$

Exercise 22. Differentiation of inverse hyperbolic functions