Module 7
Standard Integration

A. Introduction

The process of integration reverses the process of differentiation. In differentiation, if $f(x)=2x^2$ then $f'(x)=4x$. Thus the integral of $4x$ is $2x^2$, i.e. integration is the process of moving from $f'(x)$ to $f(x)$. By similar reasoning, the integral of $2t$ is t^2.

Integration is a process of summation or adding parts together and an elongated S, shown as $\int$, is used to replace the words 'the integral of'. Hence, from above, $\int 4x = 2x^2$ and $\int 2t$ is t^2.

In differentiation, the differential coefficient $\dfrac{dy}{dx}$ indicates that a function of x is being differentiated with respect to x, the dx indicating that it is 'with respect to x'. In integration the variable of integration is shown by adding d(the variable) after the function to be integrated.

Thus $\int 4x\,dx$ means 'the integral of $4x$ with respect to x',

and $\int 2t\,dt$ means 'the integral of $2t$ with respect to t'

As stated above, the differential coefficient of $2x^2$ is $4x$, hence $\int 4x\,dx = 2x^2$. However, the differential coefficient of $2x^2+7$ is also $4x$. Hence $\int 4x dx$ is also equal to $2x^2+7$. To allow for the possible presence of a constant, whenever the process of integration is performed, a constant 'c' is added to the result.

Thus $\int 4x\,dx = 2x^2 + c$ and $\int 2t\,dt = t^2 + c$

'c' is called the **arbitrary constant of integration.**

B. The general solution of integrals of the form ax^n

The general solution of integrals of the form $\int ax^n dx$, where a and n are constants is given by:

$$\int ax^n dx = \frac{ax^{n+1}}{n+1} + c$$

This rule is true when n is fractional, zero, or a positive or negative integer, with the exception of $n=-1$.
Using this rule gives:

(i) $\displaystyle\int 3x^4 dx = \frac{3x^{4+1}}{4+1} + c = \frac{3}{5}x^5 + c$

(ii) $\displaystyle\int \frac{2}{x^2} dx = \int 2x^{-2} dx = \frac{2x^{-2+1}}{-2+1} + c$

$\displaystyle = \frac{2x^{-1}}{-1} + c = \frac{-2}{x} + c,$

and

(iii) $\displaystyle\int \sqrt{x}\,dx = \int x^{1/2} dx = \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + c$

$\displaystyle = \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + c = \frac{2}{3}\sqrt{x^3} + c$

Each of these three results may be checked by differentiation.

(a) The integral of a constant k is $kx + c$. For example,

$$\int 8\,dx = 8x + c$$

(b) When a sum of several terms is integrated the result is the sum of the integrals of the separate terms. For example,

$$\int (3x + 2x^2 - 5)dx$$

$$= \int 3x\,dx + \int 2x^2 dx - \int 5\,dx$$

$$= \mathbf{\frac{3x^2}{2} + \frac{2x^3}{3} - 5x + c}$$

C. Standard Integrals

Since integration is the reverse process of differentiation the **standard integrals** listed in Table 1 may be deduced and readily checked by differentiation.

Table 1 Standard integrals

(i) $\int ax^n\,dx = \frac{ax^{n+1}}{n+1} + c$ (except when $n = -1$)

(ii) $\int \cos ax\,dx = \frac{1}{a}\sin ax + c$

(iii) $\int \sin ax\,dx = -\frac{1}{a}\cos ax + c$

(iv) $\int \sec^2 ax\,dx = \frac{1}{a}\tan ax + c$

(v) $\int \operatorname{cosec}^2 ax\,dx = -\frac{1}{a}\cot ax + c$

(vi) $\int \operatorname{cosec} ax \cot ax\,dx = -\frac{1}{a}\operatorname{cosec} ax + c$

(vii) $\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax + c$

(viii) $\int e^{ax} dx = \frac{1}{a}e^{ax} + c$

(ix) $\int \frac{1}{x}dx = \ln x + c$

Problem 1. Determine:

$$(a) \int 5x^2 dx \quad (b) \int 2t^3 dt$$

The standard integral, $\int ax^n\,dx = \frac{ax^{n+1}}{n+1} + c$

(a) When $a = 5$ and $n = 2$ then

$$\int 5x^2 dx = \frac{5x^{2+1}}{2+1} + c = \mathbf{\frac{5x^3}{3} + c}$$

(b) When $a = 2$ and $n = 3$ then

$$\int 2t^3 dt = \frac{2t^{3+1}}{3+1} + c = \frac{2t^4}{4} + c = \mathbf{\frac{1}{2}t^4 + c}$$

Each of these results may be checked by differentiating them.

Problem 2. Determine $\int \left(4 + \frac{3}{7}x - 6x^2\right) dx$

$\int \left(4 + \frac{3}{7}x - 6x^2\right) dx$ may be written as

$\int 4\,dx + \int \frac{3}{7}x\,dx - \int 6x^2\,dx$

i.e. each term is integrated separately. (This splitting up of terms only applies, however, for addition and subtraction.)

Hence

$$\int \left(4 + \frac{3}{7}x - 6x^2\right) dx$$

$$= 4x + \left(\frac{3}{7}\right)\frac{x^{1+1}}{1+1} - (6)\frac{x^{2+1}}{2+1} + c$$

$$= 4x + \left(\frac{3}{7}\right)\frac{x^2}{2} - (6)\frac{x^3}{3} + c$$

$$= \mathbf{4x + \frac{3}{14}x^2 - 2x^3 + c}$$

Note that when an integral contains more than one term there is no need to have an arbitrary constant for each; just a single constant at the end is sufficient.

Problem 3. Determine:

(a) $\int \frac{2x^3 - 3x}{4x} dx$ (b) $\int (1-t)^2 dt$

(a) Rearranging into standard integral form gives:

$$\int \frac{2x^3 - 3x}{4x} dx = \int \frac{2x^3}{4x} - \frac{3x}{4x} dx$$

$$= \int \frac{x^2}{2} - \frac{3}{4} dx = \left(\frac{1}{2}\right)\frac{x^{2+1}}{2+1} - \frac{3}{4}x + c$$

$$= \left(\frac{1}{2}\right)\frac{x^3}{3} - \frac{3}{4}x + c = \mathbf{\frac{1}{6}x^3 - \frac{3}{4}x + c}$$

(b) Rearranging $\int(1-t)^2 dt$ gives:

$$\int(1-2t+t^2)dt = t - \frac{2t^{1+1}}{1+1} + \frac{t^{2+1}}{2+1} + c$$

$$= t - \frac{2t^2}{2} + \frac{t^3}{3} + c$$

$$= \mathbf{t - t^2 + \frac{1}{3}t^3 + c}$$

This problem shows that functions often have to be rearranged into the standard form of $\int ax^n dx$ before it is possible to integrate them.

Problem 4. Determine $\int \frac{3}{x^2} dx$

$\int \frac{3}{x^2} dx = \int 3x^{-2}$. Using the standard integral, $\int ax^n dx$ when $a=3$ and $n=-2$ gives:

$$\int 3x^{-2} dx = \frac{3x^{-2+1}}{-2+1} + c = \frac{3x^{-1}}{-1} + c$$

$$= -3x^{-1} + c = \mathbf{\frac{-3}{x} + c}$$

Problem 5. Determine $\int 3\sqrt{x}\, dx$

For fractional powers it is necessary to appreciate $\sqrt[n]{a^m} = a^{\frac{m}{n}}$

$$\int 3\sqrt{x}\, dx = \int 3x^{1/2} dx = \frac{3x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + c$$

$$= \frac{3x^{\frac{3}{2}}}{\frac{3}{2}} + c = 2x^{\frac{3}{2}} + c = \mathbf{2\sqrt{x^3} + c}$$

Problem 6. Determine $\int \frac{-5}{9\sqrt[4]{t^3}} dt$

$$\int \frac{-5}{9\sqrt[4]{t^3}} dt = \int \frac{-5}{9t^{\frac{3}{4}}} dt = \int \left(-\frac{5}{9}\right) t^{-\frac{3}{4}} dt$$

$$= \left(-\frac{5}{9}\right) \frac{t^{-\frac{3}{4}+1}}{-\frac{3}{4}+1} + c$$

$$= \left(-\frac{5}{9}\right) \frac{t^{\frac{1}{4}}}{\frac{1}{4}} + c = \left(-\frac{5}{9}\right)\left(\frac{4}{1}\right) t^{1/4} + c$$

$$= \mathbf{-\frac{20}{9}\sqrt[4]{t} + c}$$

Problem 7. Determine $\int \frac{(1+\theta)^2}{\sqrt{\theta}} d\theta$

$$\int \frac{(1+\theta)^2}{\sqrt{\theta}} d\theta = \int \frac{(1+2\theta+\theta^2)}{\sqrt{\theta}} d\theta$$

$$= \int \left(\frac{1}{\theta^{\frac{1}{2}}} + \frac{2\theta}{\theta^{\frac{1}{2}}} + \frac{\theta^2}{\theta^{\frac{1}{2}}}\right) d\theta$$

$$= \int \left(\theta^{-\frac{1}{2}} + 2\theta^{1-\left(\frac{1}{2}\right)} + \theta^{2-\left(\frac{1}{2}\right)}\right) d\theta$$

$$= \int \left(\theta^{-\frac{1}{2}} + 2\theta^{\frac{1}{2}} + \theta^{\frac{3}{2}}\right) d\theta$$

$$= \frac{\theta^{\left(-\frac{1}{2}\right)+1}}{-\frac{1}{2}+1} + \frac{2\theta^{\left(\frac{1}{2}\right)+1}}{\frac{1}{2}+1} + \frac{\theta^{\left(\frac{3}{2}\right)+1}}{\frac{3}{2}+1} + c$$

$$= \frac{\theta^{\frac{1}{2}}}{\frac{1}{2}} + \frac{2\theta^{\frac{3}{2}}}{\frac{3}{2}} + \frac{\theta^{\frac{5}{2}}}{\frac{5}{2}} + c$$

$$= 2\theta^{\frac{1}{2}} + \frac{4}{3}\theta^{\frac{3}{2}} + \frac{2}{5}\theta^{\frac{5}{2}} + c$$

$$= \mathbf{2\sqrt{\theta} + \frac{4}{3}\sqrt{\theta^3} + \frac{2}{5}\sqrt{\theta^5} + c}$$

Problem 8. Determine:

(a) $\int 4\cos 3x\, dx$ (b) $\int 5\sin 2\theta\, d\theta$

(a) From Table 1 (ii),

$$\int 4\cos 3x\, dx = (4)\left(\frac{1}{3}\right)\sin 3x + c$$

$$= \mathbf{\frac{4}{3}\sin 3x + c}$$

(b) From Table 1(iii),

$$\int 5\sin 2\theta\, d\theta = (5)\left(-\frac{1}{2}\right)\cos 2\theta + c$$

$$= \mathbf{-\frac{5}{2}\cos 2\theta + c}$$

Problem 9. Determine: (a) $\int 7\sec^2 4t\,dt$

(b) $3\int \text{cosec}^2 2\theta\,d\theta$

(a) From Table 1(iv),

$$\int 7\sec^2 4t\,dt = (7)\left(\frac{1}{4}\right)\tan 4t + c$$
$$= \mathbf{\frac{7}{4}\tan 4t + c}$$

(b) From Table 1(v),

$$3\int \text{cosec}^2 2\theta\,d\theta = (3)\left(-\frac{1}{2}\right)\cot 2\theta + c$$
$$= \mathbf{-\frac{3}{2}\cot 2\theta + c}$$

Problem 10. Determine: (a) $\int 5e^{3x}\,dx$

(b) $\int \frac{2}{3e^{4t}}dt$

(a) From Table 1(viii),

$$\int 5e^{3x}\,dx = (5)\left(\frac{1}{3}\right)e^{3x} + c = \mathbf{\frac{5}{3}e^{3x} + c}$$

(b)
$$\int \frac{2}{3e^{4t}}dt = \int \frac{2}{3}e^{-4t}\,dt$$
$$= \left(\frac{2}{3}\right)\left(-\frac{1}{4}\right)e^{-4t} + c$$
$$= -\frac{1}{6}e^{-4t} + c = \mathbf{-\frac{1}{6e^{4t}} + c}$$

Problem 11. Determine:

(a) $\int \frac{3}{5x}dx$ (b) $\int \left(\frac{2m^2+1}{m}\right)dm$

(a)
$$\int \frac{3}{5x}dx = \int \left(\frac{3}{5}\right)\left(\frac{1}{x}\right)dx = \mathbf{\frac{3}{5}\ln x + c}$$

(from Table 48.1(ix))

(b)
$$\int \left(\frac{2m^2+1}{m}\right)dm = \int \left(\frac{2m^2}{m} + \frac{1}{m}\right)dm$$
$$= \int \left(2m + \frac{1}{m}\right)dm$$
$$= \frac{2m^2}{2} + \ln m + c$$
$$= \mathbf{m^2 + \ln m + c}$$

Exercise 1. Standard integrals

D. Definite Integrals

Integrals containing an arbitrary constant c in their results are called **indefinite integrals** since their precise value cannot be determined without further information. **Definite integrals** are those in which limits are applied. If an expression is written as $[x]_a^b$, 'b' is called the upper limit and 'a' the lower limit.
The operation of applying the limits is defined as: $[x]_a^b = (b) - (a)$.
The increase in the value of the integral x^2 as x increases from 1 to 3 is written as $\int_1^3 x^2\,dx$.
Applying the limits gives:

$$\int_1^3 x^2 dx = \left[\frac{x^3}{3} + c\right]_1^3 = \left(\frac{3^3}{3} + c\right) - \left(\frac{1^3}{3} + c\right)$$
$$= (9 + c) - \left(\frac{1}{3} + c\right) = \mathbf{8\frac{2}{3}}$$

Note that the 'c' term always cancels out when limits are applied and it need not be shown with definite integrals.

Problem 12. Evaluate (a) $\int_1^2 3x\,dx$

(b) $\int_{-2}^3 (4 - x^2)dx$

(a)
$$\int_1^2 3x\,dx = \left[\frac{3x^2}{2}\right]_1^2 = \left\{\frac{3}{2}(2)^2\right\} - \left\{\frac{3}{2}(1)^2\right\}$$
$$= 6 - 1\frac{1}{2} = \mathbf{4\frac{1}{2}}$$

(b)
$$\int_{-2}^3 (4 - x^2)dx = \left[4x - \frac{x^3}{3}\right]_{-2}^3$$
$$= \left\{4(3) - \frac{(3)^3}{3}\right\} - \left\{4(-2) - \frac{(-2)^3}{3}\right\}$$
$$= \{12 - 9\} - \left\{-8 - \frac{-8}{3}\right\}$$
$$= \{3\} - \left\{-5\frac{1}{3}\right\} = \mathbf{8\frac{1}{3}}$$

Problem 13. Evaluate $\int_1^4 \left(\frac{\theta+2}{\sqrt{\theta}}\right) d\theta$, taking positive square roots only

$$\int_1^4 \left(\frac{\theta+2}{\sqrt{\theta}}\right) d\theta = \int_1^4 \left(\frac{\theta}{\theta^{\frac{1}{2}}} + \frac{2}{\theta^{\frac{1}{2}}}\right) d\theta$$

$$= \int_1^4 \left(\theta^{\frac{1}{2}} + 2\theta^{-\frac{1}{2}}\right) d\theta$$

$$= \left[\frac{\theta^{\left(\frac{1}{2}\right)+1}}{\frac{1}{2}+1} + \frac{2\theta^{\left(-\frac{1}{2}\right)+1}}{-\frac{1}{2}+1}\right]_1^4$$

$$= \left[\frac{\theta^{\frac{3}{2}}}{\frac{3}{2}} + \frac{2\theta^{\frac{1}{2}}}{\frac{1}{2}}\right]_1^4 = \left[\frac{2}{3}\sqrt{\theta^3} + 4\sqrt{\theta}\right]_1^4$$

$$= \left\{\frac{2}{3}\sqrt{(4)^3} + 4\sqrt{4}\right\} - \left\{\frac{2}{3}\sqrt{(1)^3} + 4\sqrt{1}\right\}$$

$$= \left\{\frac{16}{3} + 8\right\} - \left\{\frac{2}{3} + 4\right\}$$

$$= 5\frac{1}{3} + 8 - \frac{2}{3} - 4 = \mathbf{8\frac{2}{3}}$$

Problem 14. Evaluate: $\int_0^{\pi/2} 3\sin 2x\, dx$

$$\int_0^{\frac{\pi}{2}} 3\sin 2x\, dx$$

$$= \left[(3)\left(-\frac{1}{2}\right)\cos 2x\right]_0^{\frac{\pi}{2}} = \left[-\frac{3}{2}\cos 2x\right]_0^{\frac{\pi}{2}}$$

$$= \left\{-\frac{3}{2}\cos 2\left(\frac{\pi}{2}\right)\right\} - \left\{-\frac{3}{2}\cos 2(0)\right\}$$

$$= \left\{-\frac{3}{2}\cos \pi\right\} - \left\{-\frac{3}{2}\cos 0\right\}$$

$$= \left\{-\frac{3}{2}(-1)\right\} - \left\{-\frac{3}{2}(1)\right\} = \frac{3}{2} + \frac{3}{2} = \mathbf{3}$$

Problem 15. Evaluate: $\int_1^2 4\cos 3t\, dt$

$$\int_1^2 4\cos 3t\, dt = \left[(4)\left(\frac{1}{3}\right)\sin 3t\right]_1^2 = \left[\frac{4}{3}\sin 3t\right]_1^2$$

$$= \left\{\frac{4}{3}\sin 6\right\} - \left\{\frac{4}{3}\sin 3\right\}$$

Note that limits of trigonometric functions are always expressed in radians—thus, for example, sin 6 means the sine of 6 radians $= -0.279415\ldots$

Hence $$\int_1^2 4\cos 3t\, dt = \left\{\frac{4}{3}(-0.279415\ldots)\right\} - \left\{\frac{4}{3}(-0.141120\ldots)\right\}$$

$$= (-0.37255) - (0.18816) = \mathbf{-0.5607}$$

Problem 16. Evaluate:
(a) $\int_1^2 4e^{2x}\, dx$ (b) $\int_1^4 \frac{3}{4u} du$,
each correct to 4 significant figures

(a) $$\int_1^2 4e^{2x}\, dx = \left[\frac{4}{2}e^{2x}\right]_1^2$$

$$= 2[e^{2x}]_1^2 = 2[e^4 - e^2]$$

$$= 2[54.5982 - 7.3891] = \mathbf{94.42}$$

(b) $$\int_1^4 \frac{3}{4u} du = \left[\frac{3}{4}\ln u\right]_1^4 = \frac{3}{4}[\ln 4 - \ln 1]$$

$$= \frac{3}{4}[1.3863 - 0] = \mathbf{1.040}$$

Exercise 2. Definite integrals