Module 6 - Hyperbolic Functions

A. Standard differential coefficients of hyperbolic functions

$$\frac{d}{dx}(\sinh x) = \frac{d}{dx}\left(\frac{e^x - e^{-x}}{2}\right) = \left[\frac{e^x - (-e^{-x})}{2}\right]$$

$$= \left(\frac{e^x + e^{-x}}{2}\right) = \mathbf{\cosh x}$$

If $\boldsymbol{y = \sinh ax}$, where '$a$' is a constant, then $\boldsymbol{\frac{dy}{dx} = a\cosh ax}$

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Using the quotient rule of differentiation the derivatives of $\tanh x$, $\text{sech}\, x$, $\text{cosech}\, x$ and $\coth x$ may be determined using the above results.

Problem 1. Determine the differential coefficient of: (a) $\text{th}\, x$ (b) $\text{sech}\, x$.

(a) $$\frac{d}{dx}(\text{th}\, x) = \frac{d}{dx}\left(\frac{\text{sh}\, x}{\text{ch}\, x}\right)$$

$$= \frac{(\text{ch}\, x)(\text{ch}\, x) - (\text{sh}\, x)(\text{sh}\, x)}{\text{ch}^2 x}$$

using the quotient rule

$$= \frac{\text{ch}^2 x - \text{sh}^2 x}{\text{ch}^2 x} = \frac{1}{\text{ch}^2 x} = \mathbf{sech^2}\, \boldsymbol{x}$$

(b) $$\frac{d}{dx}(\text{sech}\, x) = \frac{d}{dx}\left(\frac{1}{\text{ch}\, x}\right)$$

$$= \frac{(\text{ch}\, x)(0) - (1)(\text{sh}\, x)}{\text{ch}^2 x}$$

$$= \frac{-\text{sh}\, x}{\text{ch}^2 x} = -\left(\frac{1}{\text{ch}\, x}\right)\left(\frac{\text{sh}\, x}{\text{ch}\, x}\right)$$

$$= \mathbf{-sech}\, \boldsymbol{x}\, \mathbf{th}\, \boldsymbol{x}$$

Problem 2. Determine $\frac{dy}{d\theta}$ given (a) $y = \text{cosech}\, \theta$ (b) $y = \coth \theta$.

(a) $$\frac{d}{d\theta}(\text{cosec}\, \theta) = \frac{d}{d\theta}\left(\frac{1}{\text{sh}\, \theta}\right)$$

$$= \frac{(\text{sh}\, \theta)(0) - (1)(\text{ch}\, \theta)}{\text{sh}^2 \theta}$$

$$= \frac{-\text{ch}\, \theta}{\text{sh}^2 \theta} = -\left(\frac{1}{\text{sh}\, \theta}\right)\left(\frac{\text{ch}\, \theta}{\text{sh}\, \theta}\right)$$

$$= \mathbf{-cosech}\, \boldsymbol{\theta}\, \mathbf{coth}\, \boldsymbol{\theta}$$

(b) $$\frac{d}{d\theta}(\coth \theta) = \frac{d}{d\theta}\left(\frac{\text{ch}\, \theta}{\text{sh}\, \theta}\right)$$

$$= \frac{(\text{sh}\, \theta)(\text{sh}\, \theta) - (\text{ch}\, \theta)(\text{ch}\, \theta)}{\text{sh}^2 \theta}$$

$$= \frac{\text{sh}^2 \theta - \text{ch}^2 \theta}{\text{sh}^2 \theta} = \frac{-(\text{ch}^2 \theta - \text{sh}^2 \theta)}{\text{sh}^2 \theta}$$

$$= \frac{-1}{\text{sh}^2 \theta} = \mathbf{-cosech^2}\, \boldsymbol{\theta}$$

Summary of differential coefficients

$\boldsymbol{y}$ **or** $\boldsymbol{f(x)}$	$\boldsymbol{\frac{dy}{dx}}$ **or** $\boldsymbol{f'(x)}$
$\sinh ax$	$a\cosh ax$
$\cosh ax$	$a\sinh ax$
$\tanh ax$	$a\,\text{sech}^2 ax$
$\text{sech}\, ax$	$-a\,\text{sech}\, ax \tanh ax$
$\text{cosech}\, ax$	$-a\,\text{cosech}\, ax \coth ax$
$\coth ax$	$-a\,\text{cosech}^2 ax$

Solved problems on differentiation of hyperbolic functions

Problem 3. Differentiate the following with respect to x:

(a) $y = 4\,\text{sh}\,2x - \frac{3}{7}\text{ch}\,3x$

(b) $y = 5\,\text{th}\,\frac{x}{2} - 2\coth 4x$

(a) $y = 4\,\text{sh}\,2x - \frac{3}{7}\,\text{ch}\,3x$

$$\frac{dy}{dx} = 4(2\cosh 2x) - \frac{3}{7}(3\sinh 3x)$$

$$= \mathbf{8\cosh 2x - \frac{9}{7}\sinh 3x}$$

(b) $y = 5\,\text{th}\,\frac{x}{2} - 2\coth 4x$

$$\frac{dy}{dx} = 5\left(\frac{1}{2}\,\text{sech}^2\frac{x}{2}\right) - 2(-4\,\text{cosech}^2\,4x)$$

$$= \mathbf{\frac{5}{2}\,sech^2\frac{x}{2} + 8\,cosech^2\,4x}$$

Problem 4. Differentiate the following with respect to the variable: (a) $y = 4\sin 3t\,\text{ch}\,4t$ (b) $y = \ln(\text{sh}\,3\theta) - 4\,\text{ch}^2\,3\theta$.

(a) $y = 4\sin 3t\,\text{ch}\,4t$ (i.e. a product)

$$\frac{dy}{dx} = (4\sin 3t)(4\,\text{sh}\,4t) + (\text{ch}\,4t)(4)(3\cos 3t)$$

$$= 16\sin 3t\,\text{sh}\,4t + 12\,\text{ch}\,4t\cos 3t$$

$$= \mathbf{4(4\sin 3t\,sh\,4t + 3\cos 3t\,ch\,4t)}$$

(b) $y = \ln(\text{sh}\,3\theta) - 4\,\text{ch}^2 3\theta$

(i.e. a function of a function)

$$\frac{dy}{d\theta} = \left(\frac{1}{\text{sh}\,3\theta}\right)(3\,\text{ch}\,3\theta) - (4)(2\,\text{ch}\,3\theta)(3\,\text{sh}\,3\theta)$$

$$= 3\coth 3\theta - 24\,\text{ch}\,3\theta\,\text{sh}\,3\theta$$

$$= \mathbf{3(\coth 3\theta - 8\,ch\,3\theta\,sh\,3\theta)}$$

Problem 5. Show that the differential coefficient of

$$y = \frac{3x^2}{\text{ch}\,4x} \text{ is: } 6x\,\text{sech}\,4x\,(1 - 2x\,\text{th}\,4x)$$

$$y = \frac{3x^2}{\text{ch}\,4x} \quad \text{(i.e. a quotient)}$$

$$\frac{dy}{dx} = \frac{(\text{ch}\,4x)(6x) - (3x^2)(4\,\text{sh}\,4x)}{(\text{ch}\,4x)^2}$$

$$= \frac{6x(\text{ch}\,4x - 2x\,\text{sh}\,4x)}{\text{ch}^2\,4x}$$

$$= 6x\left[\frac{\text{ch}\,4x}{\text{ch}^2\,4x} - \frac{2x\,\text{sh}\,4x}{\text{ch}^2\,4x}\right]$$

$$= 6x\left[\frac{1}{\text{ch}\,4x} - 2x\left(\frac{\text{sh}\,4x}{\text{ch}\,4x}\right)\left(\frac{1}{\text{ch}\,4x}\right)\right]$$

$$= 6x[\text{sech}\,4x - 2x\,\text{th}\,4x\,\text{sech}\,4x]$$

$$= \mathbf{6x\,sech\,4x\,(1 - 2x\,th\,4x)}$$

Exercise 19. Differentiation of hyperbolic functions