Module 6
Logarithmic Differentiation

Logarithmic Differentiation

With certain functions containing more complicated products and quotients, differentiation is often made easier if the logarithm of the function is taken before differentiating. This technique, called **'logarithmic differentiation'** is achieved with a knowledge of (i) the laws of logarithms, (ii) the differential coefficients of logarithmic functions, and (iii) the differentiation of implicit functions.

A. Laws of logarithms

Three laws of logarithms may be expressed as:

(i) $\log(A \times B) = \log A + \log B$

(ii) $\log\left(\frac{A}{B}\right) = \log A - \log B$

(iii) $\log A^n = n \log A$

In calculus, Napierian logarithms (i.e. logarithms to a base of 'e') are invariably used. Thus for two functions $f(x)$ and $g(x)$ the laws of logarithms may be expressed as:

(i) $\ln[f(x) \cdot g(x)] = \ln f(x) + \ln g(x)$

(ii) $\ln\left(\frac{f(x)}{g(x)}\right) = \ln f(x) - \ln g(x)$

(iii) $\ln[f(x)]^n = n \ln f(x)$

Taking Napierian logarithms of both sides of the equation $y = \frac{f(x) \cdot g(x)}{h(x)}$ gives :

$$\ln y = \ln\left(\frac{f(x) \cdot g(x)}{h(x)}\right)$$

which may be simplified using the above laws of logarithms, giving;

$$\ln y = \ln f(x) + \ln g(x) - \ln h(x)$$

This latter form of the equation is often easier to differentiate.

B. Differentiation of logarithmic functions

The differential coefficient of the logarithmic function $\ln x$ is given by:

$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

More generally, it may be shown that:

$$\frac{d}{dx}[\ln f(x)] = \frac{f'(x)}{f(x)} \quad (1)$$

For example, if $y = \ln(3x^2 + 2x - 1)$ then,

$$\frac{dy}{dx} = \frac{6x + 2}{3x^2 + 2x - 1}$$

Similarly, if $y = \ln(\sin 3x)$ then

$$\frac{dy}{dx} = \frac{3\cos 3x}{\sin 3x} = 3\cot 3x.$$

Exercise 22. Differentiating logarithmic functions

C. Differentiation of further logarithmic functions

By using the function of a function rule:

$$\frac{d}{dx}(\ln y)=\left(\frac{1}{y}\right)\frac{dy}{dx} \qquad (2)$$

Differentiation of an expression such as $y=\dfrac{(1+x)^2\sqrt{(x-1)}}{x\sqrt{(x+2)}}$ may be achieved by using the product and quotient rules of differentiation; however the working would be rather complicated. With logarithmic differentiation the following procedure is adopted:

(i) Take Napierian logarithms of both sides of the equation.

$$\text{Thus } \ln y=\ln\left\{\frac{(1+x)^2\sqrt{(x-1)}}{x\sqrt{(x+2)}}\right\}$$
$$=\ln\left\{\frac{(1+x)^2(x-1)^{\frac{1}{2}}}{x(x+2)^{\frac{1}{2}}}\right\}$$

(ii) Apply the laws of logarithms.

Thus $\ln y=\ln(1+x)^2+\ln(x-1)^{\frac{1}{2}} - \ln x-\ln(x+2)^{\frac{1}{2}}$, by laws (i)

i.e. $\ln y=2\ln(1+x)+\frac{1}{2}\ln(x-1) - \ln x-\frac{1}{2}\ln(x+2)$, by law (iii)

(iii) Differentiate each term in turn with respect of x using equations (1) and (2).

$$\text{Thus } \frac{1}{y}\frac{dy}{dx}=\frac{2}{(1+x)}+\frac{\frac{1}{2}}{(x-1)}-\frac{1}{x}-\frac{\frac{1}{2}}{(x+2)}$$

(iv) Rearrange the equation to make $\frac{dy}{dx}$ the subject.

$$\text{Thus } \frac{dy}{dx}=y\left\{\frac{2}{(1+x)}+\frac{1}{2(x-1)}-\frac{1}{x}-\frac{1}{2(x+2)}\right\}$$

(v) Substitute for y in terms of x.

$$\text{Thus } \boldsymbol{\frac{dy}{dx}=\frac{(1+x)^2(\sqrt{(x-1)})}{x\sqrt{(x+2)}}\left\{\frac{2}{(1+x)}+\frac{1}{2(x-1)}-\frac{1}{x}-\frac{1}{2(x+2)}\right\}}.$$

Problem 1. Use logarithmic differentiation to differentiate $y=\dfrac{(x+1)(x-2)^3}{(x-3)}$.

Following the above procedure:

(i) Since $y=\dfrac{(x+1)(x-2)^3}{(x-3)}$

then $\ln y=\ln\left\{\dfrac{(x+1)(x-2)^3}{(x-3)}\right\}$

(ii) $\ln y=\ln(x+1)+\ln(x-2)^3-\ln(x-3)$, by laws (i) and (ii)

i.e. $\ln y=\ln(x+1)+3\ln(x-2)-\ln(x-3)$, by law (iii)

(iii) Differentiating with respect to x gives:

$$\frac{1}{y}\frac{dy}{dx}=\frac{1}{(x+1)}+\frac{3}{(x-2)}-\frac{1}{(x-3)}$$

by using equations (1) and (2).

(iv) Rearranging gives:

$$\frac{dy}{dx} = y\left\{\frac{1}{(x+1)} + \frac{3}{(x-2)} - \frac{1}{(x-3)}\right\}$$

(v) Substituting for y gives:

$$\boldsymbol{\frac{dy}{dx} = \frac{(x+1)(x-2)^3}{(x-3)}\left\{\frac{1}{(x+1)} + \frac{3}{(x-2)} - \frac{1}{(x-3)}\right\}}$$

Problem 2. Differentiate $y = \dfrac{\sqrt{(x-2)^3}}{(x+1)^2(2x-1)}$ with respect to x and evalualte $\dfrac{dy}{dx}$ when $x = 3$.

Using logarithmic differentiation and following the above procedure:

(i) Since $y = \dfrac{\sqrt{(x-2)^3}}{(x+1)^2(2x-1)}$

then $\ln y = \ln\left\{\dfrac{\sqrt{(x-2)^3}}{(x+1)^2(2x-1)}\right\}$

$= \ln\left\{\dfrac{(x-2)^{\frac{3}{2}}}{(x+1)^2(2x-1)}\right\}$

(ii) $\ln y = \ln(x-2)^{\frac{3}{2}} - \ln(x+1)^2 - \ln(2x-1)$

i.e. $\ln y = \frac{3}{2}\ln(x-2) - 2\ln(x+1) - \ln(2x-1)$

(iii) $$\frac{1}{y}\frac{dy}{dx} = \frac{\frac{3}{2}}{(x-2)} - \frac{2}{(x+1)} - \frac{2}{(2x-1)}$$

(iv) $$\frac{dy}{dx} = y\left\{\frac{3}{2(x-2)} - \frac{2}{(x+1)} - \frac{2}{(2x-1)}\right\}$$

(v) $$\boldsymbol{\frac{dy}{dx} = \frac{\sqrt{(x-2)^3}}{(x+1)^2(2x-1)}\left\{\frac{3}{2(x-2)} - \frac{2}{(x+1)} - \frac{2}{(2x-1)}\right\}}$$

When $x = 3$, $$\frac{dy}{dx} = \frac{\sqrt{(1)^3}}{(4)^2(5)}\left(\frac{3}{2} - \frac{2}{4} - \frac{2}{5}\right)$$

$$= \pm\frac{1}{80}\left(\frac{3}{5}\right) = \boldsymbol{\pm\frac{3}{400}} \text{ or } \boldsymbol{\pm 0.0075}$$

Problem 3. Given $y = \dfrac{3e^{2\theta}\sec 2\theta}{\sqrt{(\theta-2)}}$ determine $\dfrac{dy}{d\theta}$

Using logarithmic differentiation and following the procedure:

(i) Since $y = \dfrac{3e^{2\theta}\sec 2\theta}{\sqrt{(\theta-2)}}$

then $\ln y = \ln\left\{\dfrac{3e^{2\theta}\sec 2\theta}{\sqrt{(\theta-2)}}\right\}$

$= \ln\left\{\dfrac{3e^{2\theta}\sec 2\theta}{(\theta-2)^{\frac{1}{2}}}\right\}$

(ii) $\ln y = \ln 3e^{2\theta} + \ln\sec 2\theta - \ln(\theta-2)^{\frac{1}{2}}$

i.e. $\ln y = \ln 3 + \ln e^{2\theta} + \ln\sec 2\theta - \frac{1}{2}\ln(\theta-2)$

i.e. $\ln y = \ln 3 + 2\theta + \ln\sec 2\theta - \frac{1}{2}\ln(\theta-2)$

(iii) Differentiating with respect to θ gives:

$$\frac{1}{y}\frac{dy}{d\theta} = 0 + 2 + \frac{2\sec 2\theta\tan 2\theta}{\sec 2\theta} - \frac{\frac{1}{2}}{(\theta-2)}$$

from equations (1) and (2).

(iv) Rearranging gives:

$$\frac{dy}{d\theta} = y\left\{2 + 2\tan 2\theta - \frac{1}{2(\theta-2)}\right\}$$

(v) Substituting for y gives:

$$\boldsymbol{\frac{dy}{d\theta} = \frac{3e^{2\theta}\sec 2\theta}{\sqrt{(\theta-2)}}\left\{2 + 2\tan 2\theta - \frac{1}{2(\theta-2)}\right\}}$$

Problem 4. Differentiate $y = \dfrac{x^3\ln 2x}{e^x\sin x}$ with respect to x

Using logarithmic differentiation and following the procedure gives:

(i) $\ln y = \ln\left\{\dfrac{x^3\ln 2x}{e^x\sin x}\right\}$

(ii) $\ln y = \ln x^3 + \ln(\ln 2x) - \ln(e^x) - \ln(\sin x)$

i.e. $\ln y = 3\ln x + \ln(\ln 2x) - x - \ln(\sin x)$

(iii) $$\frac{1}{y}\frac{dy}{dx} = \frac{3}{x} + \frac{\frac{1}{x}}{\ln 2x} - 1 - \frac{\cos x}{\sin x}$$

(iv) $\dfrac{dy}{dx}=y\left\{\dfrac{3}{x}+\dfrac{1}{x\ln 2x}-1-\cot x\right\}$

(v) $\boldsymbol{\dfrac{dy}{dx}=\dfrac{x^3\ln 2x}{e^x\sin x}\left\{\dfrac{3}{x}+\dfrac{1}{x\ln 2x}-1-\cot x\right\}}$

Exercise 23. Differentiating logarithmic functions

D. Differentiation of [$f(x)$]^x

Whenever an expression to be differentiated contains a term raised to a power which is itself a function of the variable, then logarithmic differentiation must be used. For example, the differentiation of expressions such as x^x, $(x+2)^x$, $\sqrt[x]{(x-1)}$ and x^{3x+2} can only be achieved using logarithmic differentiation.

Problem 5. Determine $\dfrac{dy}{dx}$ given $y=x^x$.

Taking Napierian logarithms of both sides of $y=x^x$ gives:

$\ln\ y=\ln x^x=x\ln x$,

Differentiating both sides with respect to x gives:

$\dfrac{1}{y}\dfrac{dy}{dx}=(x)\left(\dfrac{1}{x}\right)+(\ln x)(1)$, using the product rule

i.e. $\dfrac{1}{y}\dfrac{dy}{dx}=1+\ln x$

from which, $\dfrac{dy}{dx}=y(1+\ln x)$

i.e. $\boldsymbol{\dfrac{dy}{dx}=x^x(1+\ln x)}$

Problem 6. Evaluate $\dfrac{dy}{dx}$ when $x=-1$ given $y=(x+2)^x$

Taking Napierian logarithms of both sides of $y=(x+2)^x$ gives:

$\ln\ y=\ln(x+2)^x=x\ln(x+2)$

Differentiating both sides with respect to x gives:

$$\frac{1}{y}\frac{dy}{dx} = (x)\left(\frac{1}{x+2}\right) + [\ln(x+2)](1),$$

by the product rule.

$$\text{Hence} \quad \frac{dy}{dx} = y\left(\frac{x}{x+2} + \ln(x+2)\right)$$

$$= \mathbf{(x+2)^x\left\{\frac{x}{x+2} + \ln(x+2)\right\}}$$

$$\text{When } x=-1,\ \frac{dy}{dx} = (1)^{-1}\left(\frac{-1}{1} + \ln 1\right)$$

$$= (+1)(-1) = \mathbf{-1}$$

Problem 7. Determine (a) the differential coefficient of $y = \sqrt[x]{(x-1)}$ and (b) evaluate $\dfrac{dy}{dx}$ when $x=2$.

(a) $y = \sqrt[x]{(x-1)} = (x-1)^{\frac{1}{x}}$, since by the laws of indices $\sqrt[n]{a^m} = a^{\frac{m}{n}}$

Taking Napierian logarithms of both sides gives:

$$\ln y = \ln(x-1)^{\frac{1}{x}} = \frac{1}{x}\ln(x-1),$$

Differentiating each side with respect to x gives:

$$\frac{1}{y}\frac{dy}{dx} = \left(\frac{1}{x}\right)\left(\frac{1}{x-1}\right) + [\ln(x-1)]\left(\frac{-1}{x^2}\right)$$

by the product rule.

$$\text{Hence } \frac{dy}{dx} = y\left\{\frac{1}{x(x-1)} - \frac{\ln(x-1)}{x^2}\right\}$$

$$\text{i.e. } \mathbf{\frac{dy}{dx} = \sqrt[x]{(x-1)}\left\{\frac{1}{x(x-1)} - \frac{\ln(x-1)}{x^2}\right\}}$$

(b) When $x=2$, $\quad \dfrac{dy}{dx} = \sqrt[2]{(1)}\left\{\dfrac{1}{2(1)} - \dfrac{\ln(1)}{4}\right\}$

$$= \pm 1\left\{\frac{1}{2} - 0\right\} = \pm\,\mathbf{\frac{1}{2}}$$

Problem 8. Differentiate x^{3x+2} with respect to x

Let $y = x^{3x+2}$

Taking Napierian logarithms of both sides gives:

$$\ln y = \ln x^{3x+2}$$

i.e. $\ln y = (3x+2)\ln x$, by law (iii)

Differentiating each term with respect to x gives:

$$\frac{1}{y}\frac{dy}{dx} = (3x+2)\left(\frac{1}{x}\right) + (\ln x)(3),$$

by the product rule.

$$\text{Hence} \quad \frac{dy}{dx} = y\left\{\frac{3x+2}{x} + 3\ln x\right\}$$

$$= x^{3x+2}\left\{\frac{3x+2}{x} + 3\ln x\right\}$$

$$= \mathbf{x^{3x+2}\left\{3 + \frac{2}{x} + 3\ln x\right\}}$$

Exercise 24. differentiating [f(x)]x type functions