Module 5
Implicit Functions

Implicit functions

When an equation can be written in the form $y = f(x)$ it is said to be an **explicit function** of x. Examples of explicit functions include

$$y = 2x^3 - 3x + 4, \quad y = 2x \ln x$$

and $$y = \frac{3e^x}{\cos x}$$

In these examples y may be differentiated with respect to x by using standard derivatives, the product rule and the quotient rule of differentiation respectively.

Sometimes with equations involving, say, y and x, it is impossible to make y the subject of the formula. The equation is then called an **implicit function** and examples of such functions include $y^3 + 2x^2 = y^2 - x$ and $\sin y = x^2 + 2xy$.

A. Differentiating implicit functions

It is possible to **differentiate an implicit function** by using the **function of a function rule**, which may be stated as

$$\frac{du}{dx} = \frac{du}{dy} \times \frac{dy}{dx}$$

Thus, to differentiate y^3 with respect to x, the substitution $u = y^3$ is made, from which, $\frac{du}{dy} = 3y^2$.

Hence, $\frac{d}{dx}(y^3) = (3y^2) \times \frac{dy}{dx}$, by the function of a function rule.

A simple rule for differentiating an implicit function is summarised as:

$$\frac{d}{dx}[f(y)] = \frac{d}{dy}[f(y)] \times \frac{dy}{dx} \qquad (1)$$

Problem 1. Differentiate the following functions with respect to x:

(a) $2y^4$ (b) $\sin 3t$

(a) Let $u = 2y^4$, then, by the function of a function rule:

$$\frac{du}{dx} = \frac{du}{dy} \times \frac{dy}{dx} = \frac{d}{dy}(2y^4) \times \frac{dy}{dx}$$
$$= \mathbf{8y^3 \frac{dy}{dx}}$$

(b) Let $u = \sin 3t$, then, by the function of a function rule:

$$\frac{du}{dx} = \frac{du}{dt} \times \frac{dt}{dx} = \frac{d}{dt}(\sin 3t) \times \frac{dt}{dx}$$
$$= \mathbf{3\cos 3t \frac{dy}{dx}}$$

Problem 2. Differentiate the following functions with respect to x:

(a) $4 \ln 5y$ (b) $\frac{1}{5}e^{3\theta - 2}$

(a) Let $u = 4 \ln 5y$, then, by the function of a function rule:

$$\frac{du}{dx} = \frac{du}{dy} \times \frac{dy}{dx} = \frac{d}{dy}(4\ln 5y) \times \frac{dy}{dx}$$
$$= \mathbf{\frac{4}{y}\frac{dy}{dx}}$$

(b) Let $u = \frac{1}{5}e^{3\theta-2}$, then, by the function of a function rule:

$$\frac{du}{dx} = \frac{du}{d\theta} \times \frac{d\theta}{dx} = \frac{d}{d\theta}\left(\frac{1}{5}e^{3\theta-2}\right) \times \frac{d\theta}{dx}$$

$$= \mathbf{\frac{3}{5}e^{3\theta-2}\frac{d\theta}{dx}}$$

Exercise 19. Differentiating implicit functions

B. Differentiating implicit functions containing products & quotients

The product and quotient rules of differentiation must be applied when differentiating functions containing products and quotients of two variables.

For example, $\frac{d}{dx}(x^2y) = (x^2)\frac{d}{dx}(y) + (y)\frac{d}{dx}(x^2)$, by the product rule

$$= (x^2)\left(1\frac{dy}{dx}\right) + y(2x),$$ by using equation (1)

$$= \mathbf{x^2\frac{dy}{dx} + 2xy}$$

Problem 3. Determine $\frac{d}{dx}(2x^3y^2)$

In the product rule of differentiation let $u = 2x^3$ and $v = y^2$.

Thus $\frac{d}{dx}(2x^3y^2) = (2x^3)\frac{d}{dx}(y^2) + (y^2)\frac{d}{dx}(2x^3)$

$$= (2x^3)\left(2y\frac{dy}{dx}\right) + (y^2)(6x^2)$$

$$= 4x^3y\frac{dy}{dx} + 6x^2y^2$$

$$= \mathbf{2x^2y\left(2x\frac{dy}{dx} + 3y\right)}$$

Problem 4. Find $\frac{d}{dx}\left(\frac{3y}{2x}\right)$

In the quotient rule of differentiation let $u = 3y$ and $v = 2x$.

$$\text{Thus } \frac{d}{dx}\left(\frac{3y}{2x}\right) = \frac{(2x)\frac{d}{dx}(3y) - (3y)\frac{d}{dx}(2x)}{(2x)^2}$$

$$= \frac{(2x)\left(3\frac{dy}{dx}\right) - (3y)(2)}{4x^2}$$

$$= \frac{6x\frac{dy}{dx} - 6y}{4x^2} = \mathbf{\frac{3}{2x^2}\left(x\frac{dy}{dx} - y\right)}$$

Problem 5. Differentiate $z = x^2 + 3x\cos 3y$ with respect to y.

$$\frac{dz}{dy} = \frac{d}{dy}(x^2) + \frac{d}{dy}(3x\cos 3y)$$
$$= 2x\frac{dx}{dy} + \left[(3x)(-3\sin 3y) + (\cos 3y)\left(3\frac{dx}{dy}\right)\right]$$
$$= \mathbf{2x\frac{dx}{dy} - 9x\sin 3y + 3\cos 3y\frac{dx}{dy}}$$

Exercise 20. Differentiating implicit functions involving products and quotients

1. Determine $\frac{d}{dx}(3x^2y^3)$ $\left[3xy^2\left(3x\frac{dy}{dx} + 2y\right)\right]$
2. Find $\frac{d}{dx}\left(\frac{2y}{5x}\right)$ $\left[\frac{2}{5x^2}\left(x\frac{dy}{dx} - y\right)\right]$
3. Determine $\frac{d}{du}\left(\frac{3u}{4v}\right)$ $\left[\frac{3}{4v^2}\left(v - u\frac{dv}{du}\right)\right]$
4. Given $z = 3\sqrt{y}\cos 3x$ find $\frac{dz}{dx}$ $\left[3\left(\frac{\cos 3x}{2\sqrt{y}}\right)\frac{dy}{dx} - 9\sqrt{y}\sin 3x\right]$
5. Determine $\frac{dz}{dy}$ given $z = 2x^3\ln y$ $\left[2x^2\left(\frac{x}{y} + 3\ln y\frac{dx}{dy}\right)\right]$

An implicit function such as $3x^2 + y^2 - 5x + y = 2$, may be differentiated term by term with respect to x. This gives:

$$\frac{d}{dx}(3x^2) + \frac{d}{dx}(y^2) - \frac{d}{dx}(5x) + \frac{d}{dx}(y) = \frac{d}{dx}(2)$$

i.e. $$6x + 2y\frac{dy}{dx} - 5 + 1\frac{dy}{dx} = 0,$$

using equation (1) and standard derivatives.

An expression for the derivative $\frac{dy}{dx}$ in terms of x and y may be obtained by rearranging this latter equation. Thus:

$$(2y+1)\frac{dy}{dx} = 5 - 6x$$

from which, $$\mathbf{\frac{dy}{dx} = \frac{5-6x}{2y+1}}$$

Problem 6. Given $2y^2 - 5x^4 - 2 - 7y^3 = 0$, determine $\frac{dy}{dx}$

Each term in turn is differentiated with respect to x:

Hence $$\frac{d}{dx}(2y^2) - \frac{d}{dx}(5x^4) - \frac{d}{dx}(2) - \frac{d}{dx}(7y^3) = \frac{d}{dx}(0)$$

i.e. $$4y\frac{dy}{dx} - 20x^3 - 0 - 21y^2\frac{dy}{dx} = 0$$

Rearranging gives:

$$(4y - 21y^2)\frac{dy}{dx} = 20x^3$$

i.e. $$\mathbf{\frac{dy}{dx} = \frac{20x^3}{(4y - 21y^2)}}$$

Problem 7. Determine the values of $\frac{dy}{dx}$ when $x = 4$ given that $x^2 + y^2 = 25$.

Differentiating each term in turn with respect to x gives:

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = \frac{d}{dx}(25)$$

i.e. $$2x + 2y\frac{dy}{dx} = 0$$

Hence $$\frac{dy}{dx} = -\frac{2x}{2y} = -\frac{x}{y}$$

Since $x^2 + y^2 = 25$, when $x = 4$, $y = \sqrt{(25 - 4^2)} = \pm 3$

Thus when $x = 4$ and $y = \pm 3$, $\mathbf{\frac{dy}{dx}} = -\frac{4}{\pm 3} = \mathbf{\pm\frac{4}{3}}$

$x^2 + y^2 = 25$ is the equation of a circle, centre at the origin and radius 5, as shown in Fig. 22. At $x = 4$, the two gradients are shown.

Above, $x^2 + y^2 = 25$ was differentiated implicitly; actually, the equation could be transposed to

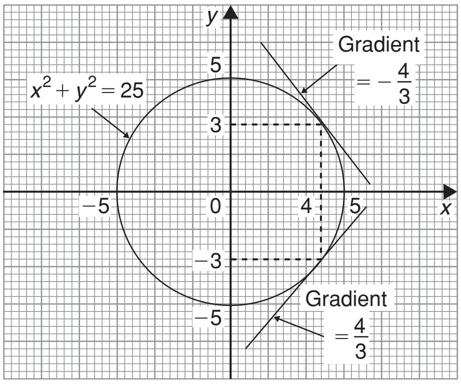

Figure 22

$y=\sqrt{(25-x^2)}$ and differentiated using the function of a function rule. This gives

$$\frac{dy}{dx}=\frac{1}{2}(25-x^2)^{\frac{-1}{2}}(-2x)=-\frac{x}{\sqrt{(25-x^2)}}$$

and when $x=4$, $\frac{dy}{dx}=-\frac{4}{\sqrt{(25-4^2)}}=\pm\frac{4}{3}$ as obtained above.

Problem 8.
(a) Find $\frac{dy}{dx}$ in terms of x and y given

$4x^2+2xy^3-5y^2=0$

(b) Evalate $\frac{dy}{dx}$ when $x=1$ and $y=2$

(a) Differentiating each term in turn with respect to x gives:

$$\frac{d}{dx}(4x^2)+\frac{d}{dx}(2xy^3)-\frac{d}{dx}(5y^2)=\frac{d}{dx}(0)$$

i.e. $$8x+\left[(2x)\left(3y^2\frac{dy}{dx}\right)+(y^3)(2)\right]-10y\frac{dy}{dx}=0$$

i.e. $$8x+6xy^2\frac{dy}{dx}+2y^3-10y\frac{dy}{dx}=0$$

Rearranging gives:

$$8x+2y^3=(10y-6xy^2)\frac{dy}{dx}$$

and $$\frac{\mathbf{dy}}{\mathbf{dx}}=\frac{8x+2y^3}{10y-6xy^2}=\frac{\mathbf{4x+y^3}}{\mathbf{y(5-3xy)}}$$

(b) When $x=1$ and $y=2$,

$$\frac{\mathbf{dy}}{\mathbf{dx}}=\frac{4(1)+(2)^3}{2[5-(3)(1)(2)]}=\frac{12}{-2}=\mathbf{-6}$$

Problem 9. Find the gradients of the tangents drawn to the circle $x^2+y^2-2x-2y=3$ at $x=2$.

The gradient of the tangent is given by $\frac{dy}{dx}$

Differentiating each term in turn with respect to x gives:

$$\frac{d}{dx}(x^2)+\frac{d}{dx}(y^2)-\frac{d}{dx}(2x)-\frac{d}{dx}(2y)=\frac{d}{dx}(3)$$

i.e. $$2x+2y\frac{dy}{dx}-2-2\frac{dy}{dx}=0$$

Hence $$(2y-2)\frac{dy}{dx}=2-2x,$$

from which $$\frac{dy}{dx}=\frac{2-2x}{2y-2}=\frac{1-x}{y-1}$$

The value of y when $x=2$ is determined from the original equation

Hence $(2)^2+y^2-2(2)-2y=3$

i.e. $4+y^2-4-2y=3$

or $y^2-2y-3=0$

Factorising gives: $(y+1)(y-3)=0$, from which $y=-1$ or $y=3$

When $x=2$ and $y=-1$,

$$\frac{dy}{dx}=\frac{1-x}{y-1}=\frac{1-2}{-1-1}=\frac{-1}{-2}=\frac{1}{2}$$

When $x=2$ and $y=3$,

$$\frac{dy}{dx}=\frac{1-2}{3-1}=\frac{-1}{2}$$

Hence the gradients of the tangents are ± 1/2

Problem 10. Pressure p and volume v of a gas are related by the law $pv^\gamma=k$, w h e r e γ and k are constants. Show that the rate of change of pressure $\frac{dp}{dt}=-\gamma\frac{p}{v}\frac{dv}{dt}$

Since $pv^{\gamma} = k$, then $p = \dfrac{k}{v^{\gamma}} = kv^{-\gamma}$

$$\frac{dp}{dt} = \frac{dp}{dv} \times \frac{dv}{dt}$$

by the function of a function rule

$$\frac{dp}{dv} = \frac{d}{dv}(kv^{-\gamma})$$

$$= -\gamma kv^{-\gamma-1} = \frac{-\gamma k}{v^{\gamma+1}}$$

$$\frac{dp}{dt} = \frac{-\gamma k}{v^{\gamma+1}} \times \frac{dv}{dt}$$

Since $\quad k = pv^{\gamma}$

$$\frac{dp}{dt} = \frac{-\gamma (pv^{\gamma})}{v^{r+1}} \frac{dv}{dt} = \frac{-\gamma pv^{\gamma}}{v^{\gamma} v^{1}} \frac{dv}{dt}$$

i.e. $\quad \boldsymbol{\dfrac{dp}{dt} = -\gamma \dfrac{p}{v} \dfrac{dv}{dt}}$

Exercise 21. Implicit differentiation