Module 4 - Implicit Functions

Implicit functions

When an equation can be written in the form $y = f(x)$ it is said to be an **explicit function** of x. Examples of explicit functions include

$$y = 2x^3 - 3x + 4, \quad y = 2x \ln x$$

and $\quad y = \dfrac{3e^x}{\cos x}$

In these examples y may be differentiated with respect to x by using standard derivatives, the product rule and the quotient rule of differentiation respectively.

Sometimes with equations involving, say, y and x, it is impossible to make y the subject of the formula. The equation is then called an **implicit function** and examples of such functions include $y^3 + 2x^2 = y^2 - x$ and $\sin y = x^2 + 2xy$.

A. Differentiating Implicit Functions

It is possible to **differentiate an implicit function** by using the **function of a function rule**, which may be stated as

$$\frac{du}{dx} = \frac{du}{dy} \times \frac{dy}{dx}$$

Thus, to differentiate y^3 with respect to x, the substitution $u = y^3$ is made, from which, $\dfrac{du}{dy} = 3y^2$. Hence, $\dfrac{d}{dx}(y^3) = (3y^2) \times \dfrac{dy}{dx}$, by the function of a function rule.

A simple rule for differentiating an implicit function is summarised as:

$$\boldsymbol{\frac{d}{dx}[f(y)] = \frac{d}{dy}[f(y)] \times \frac{dy}{dx}} \qquad (1)$$

Problem 1. Differentiate the following functions with respect to x:

(a) $2y^4$ (b) $\sin 3t$.

(a) Let $u = 2y^4$, then, by the function of a function rule:

$$\frac{du}{dx} = \frac{du}{dy} \times \frac{dy}{dx} = \frac{d}{dy}(2y^4) \times \frac{dy}{dx}$$
$$= \mathbf{8y^3 \frac{dy}{dx}}$$

(b) Let $u = \sin 3t$, then, by the function of a function rule:

$$\frac{du}{dx} = \frac{du}{dt} \times \frac{dt}{dx} = \frac{d}{dt}(\sin 3t) \times \frac{dt}{dx}$$
$$= \mathbf{3 \cos 3t \frac{dt}{dx}}$$

Problem 2. Differentiate the following functions with respect to x:

(a) $4 \ln 5y$ (b) $\dfrac{1}{5}e^{3\theta - 2}$

(a) Let $u = 4 \ln 5y$, then, by the function of a function rule:

$$\frac{du}{dx} = \frac{du}{dy} \times \frac{dy}{dx} = \frac{d}{dy}(4 \ln 5y) \times \frac{dy}{dx}$$
$$= \mathbf{\frac{4}{y} \frac{dy}{dx}}$$

(b) Let $u = \dfrac{1}{5}e^{3\theta - 2}$, then, by the function of a function rule:

$$\frac{du}{dx} = \frac{du}{d\theta} \times \frac{d\theta}{dx} = \frac{d}{d\theta}\left(\frac{1}{5}e^{3\theta - 2}\right) \times \frac{d\theta}{dx}$$
$$= \mathbf{\frac{3}{5}e^{3\theta - 2} \frac{d\theta}{dx}}$$

Exercise 14. Differentiation of implicit functions

B. Differentiating Implicit Functions containing Products and Quotients

The product and quotient rules of differentiation must be applied when differentiating functions containing products and quotients of two variables.

For example, $\frac{d}{dx}(x^2y) = (x^2)\frac{d}{dx}(y) + (y)\frac{d}{dx}(x^2)$, by the product rule

$= (x^2)\left(1\frac{dy}{dx}\right) + y(2x)$, by using equation (1)

$$= \mathbf{x^2\frac{dy}{dx} + 2xy}$$

Problem 3. Determine $\frac{d}{dx}(2x^3y^2)$.

In the product rule of differentiation let $u = 2x^3$ and $v = y^2$.

Thus $\frac{d}{dx}(2x^3y^2) = (2x^3)\frac{d}{dx}(y^2) + (y^2)\frac{d}{dx}(2x^3)$

$$= (2x^3)\left(2y\frac{dy}{dx}\right) + (y^2)(6x^2)$$

$$= 4x^3y\frac{dy}{dx} + 6x^2y^2$$

$$= \mathbf{2x^2y\left(2x\frac{dy}{dx} + 3y\right)}$$

Problem 4. Find $\frac{d}{dx}\left(\frac{3y}{2x}\right)$.

In the quotient rule of differentiation let $u = 3y$ and $v = 2x$.

Thus $\frac{d}{dx}\left(\frac{3y}{2x}\right) = \frac{(2x)\frac{d}{dx}(3y) - (3y)\frac{d}{dx}(2x)}{(2x)^2}$

$$= \frac{(2x)\left(3\frac{dy}{dx}\right) - (3y)(2)}{4x^2}$$

$$= \frac{6x\frac{dy}{dx} - 6y}{4x^2} = \mathbf{\frac{3}{2x^2}\left(x\frac{dy}{dx} - y\right)}$$

Problem 5. Differentiate $z = x^2 + 3x\cos 3y$ with respect to y.

$$\frac{dz}{dy} = \frac{d}{dy}(x^2) + \frac{d}{dy}(3x\cos 3y)$$
$$= 2x\frac{dx}{dy} + \left[(3x)(-3\sin 3y) + (\cos 3y)\left(3\frac{dx}{dy}\right)\right]$$
$$= \mathbf{2x\frac{dx}{dy} - 9x\sin 3y + 3\cos 3y\,\frac{dx}{dy}}$$

Exercise 15. Differentiating implicit functions involving products and quotients

More Implicit Differentiation

An implicit function such as $3x^2 + y^2 - 5x + y = 2$, may be differentiated term by term with respect to x. This gives:

$$\frac{d}{dx}(3x^2) + \frac{d}{dx}(y^2) - \frac{d}{dx}(5x) + \frac{d}{dx}(y) = \frac{d}{dx}(2)$$

i.e. $$6x + 2y\frac{dy}{dx} - 5 + 1\frac{dy}{dx} = 0,$$

using equation (1) and standard derivatives.

An expression for the derivative $\frac{dy}{dx}$ in terms of x and y may be obtained by rearranging this latter equation. Thus:

$$(2y+1)\frac{dy}{dx} = 5 - 6x$$

from which, $$\mathbf{\frac{dy}{dx} = \frac{5-6x}{2y+1}}$$

Problem 6. Given $2y^2 - 5x^4 - 2 - 7y^3 = 0$, determine $\frac{dy}{dx}$.

Each term in turn is differentiated with respect to x:

Hence $$\frac{d}{dx}(2y^2) - \frac{d}{dx}(5x^4) - \frac{d}{dx}(2) - \frac{d}{dx}(7y^3) = \frac{d}{dx}(0)$$

i.e. $$4y\frac{dy}{dx} - 20x^3 - 0 - 21y^2\frac{dy}{dx} = 0$$

Rearranging gives:

$$(4y - 21y^2)\frac{dy}{dx} = 20x^3$$

i.e. $$\mathbf{\frac{dy}{dx} = \frac{20x^3}{(4y - 21y^2)}}$$

Problem 7. Determine the values of $\frac{dy}{dx}$ when $x = 4$ given that $x^2 + y^2 = 25$.

Differentiating each term in turn with respect to x gives:

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = \frac{d}{dx}(25)$$

i.e. $$2x + 2y\frac{dy}{dx} = 0$$

Hence $$\frac{dy}{dx} = -\frac{2x}{2y} = -\frac{x}{y}$$

Since $x^2 + y^2 = 25$, when $x = 4$, $y = \sqrt{(25 - 4^2)} = \pm 3$

Thus when $x = 4$ and $y = \pm 3$, $\mathbf{\frac{dy}{dx}} = -\frac{4}{\pm 3} = \mathbf{\pm\frac{4}{3}}$

$x^2 + y^2 = 25$ is the equation of a circle, centre at the origin and radius 5, as shown in Fig. 19. At $x = 4$, the two gradients are shown.

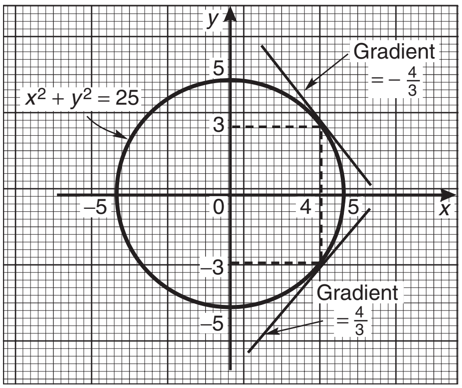

Figure 19

Above, $x^2 + y^2 = 25$ was differentiated implicitly; actually, the equation could be transposed to $y = \sqrt{(25 - x^2)}$ and differentiated using the function of a function rule. This gives

$$\frac{dy}{dx} = \frac{1}{2}(25 - x^2)^{\frac{-1}{2}}(-2x) = -\frac{x}{\sqrt{(25 - x^2)}}$$

and when $x = 4$, $\frac{dy}{dx} = -\frac{4}{\sqrt{(25 - 4^2)}} = \pm\frac{4}{3}$ as obtained above.

Problem 8.

(a) Find $\frac{dy}{dx}$ in terms of x and y given $4x^2 + 2xy^3 - 5y^2 = 0$.

(b) Evaluate $\frac{dy}{dx}$ when $x = 1$ and $y = 2$.

(a) Differentiating each term in turn with respect to x gives:

$$\frac{d}{dx}(4x^2) + \frac{d}{dx}(2xy^3) - \frac{d}{dx}(5y^2) = \frac{d}{dx}(0)$$

i.e. $$8x + \left[(2x)\left(3y^2\frac{dy}{dx}\right) + (y^3)(2)\right] - 10y\frac{dy}{dx} = 0$$

i.e. $$8x + 6xy^2\frac{dy}{dx} + 2y^3 - 10y\frac{dy}{dx} = 0$$

Rearranging gives:

$$8x + 2y^3 = (10y - 6xy^2)\frac{dy}{dx}$$

and $$\mathbf{\frac{dy}{dx}} = \frac{8x + 2y^3}{10y - 6xy^2} = \mathbf{\frac{4x + y^3}{y(5 - 3xy)}}$$

(b) When $x = 1$ and $y = 2$,

$$\mathbf{\frac{dy}{dx}} = \frac{4(1) + (2)^3}{2[5 - (3)(1)(2)]} = \frac{12}{-2} = \mathbf{-6}$$

Problem 9. Find the gradients of the tangents drawn to the circle $x^2 + y^2 - 2x - 2y = 3$ at $x = 2$.

The gradient of the tangent is given by $\frac{dy}{dx}$

Differentiating each term in turn with respect to x gives:

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) - \frac{d}{dx}(2x) - \frac{d}{dx}(2y) = \frac{d}{dx}(3)$$

i.e. $$2x + 2y\frac{dy}{dx} - 2 - 2\frac{dy}{dx} = 0$$

Hence $$(2y - 2)\frac{dy}{dx} = 2 - 2x,$$

from which $$\frac{dy}{dx} = \frac{2 - 2x}{2y - 2} = \frac{1 - x}{y - 1}$$

The value of y when $x = 2$ is determined from the original equation

Hence $(2)^2 + y^2 - 2(2) - 2y = 3$

i.e. $4 + y^2 - 4 - 2y = 3$

or $y^2 - 2y - 3 = 0$

Factorising gives: $(y + 1)(y - 3) = 0$, from which $y = -1$ or $y = 3$

When $x = 2$ and $y = -1$,

$$\frac{dy}{dx} = \frac{1 - x}{y - 1} = \frac{1 - 2}{-1 - 1} = \frac{-1}{-2} = \frac{1}{2}$$

When $x = 2$ and $y = 3$,

$$\frac{dy}{dx} = \frac{1 - 2}{3 - 1} = \frac{-1}{2}$$

Hence the gradients of the tangents are $\mathbf{\pm\frac{1}{2}}$

Exercise 16. Implicit differentiation

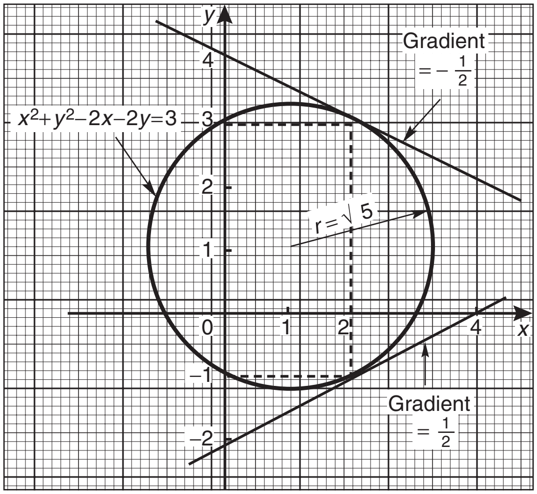

Figure 20

The circle having the given equation has its centre at (1, 1) and radius √ and is shown. in Fig. 20 with the two gradients of the tangents.

Problem 10. Pressure p and volume v of a gas are related by the law $pv^{\gamma}=k$, where γ and k are constants. Show that the rate of change of pressure $\frac{dp}{dt}=-\gamma\frac{p}{v}\frac{dv}{dt}$

Since $pv^{\gamma}=k$, then $p=\frac{k}{v^{\gamma}}=kv^{-\gamma}$

$$\frac{dp}{dt}=\frac{dp}{dv}\times\frac{dv}{dt}$$

by the function of a function rule

$$\frac{dp}{dv}=\frac{d}{dv}(kv^{-\gamma})$$

$$=-\gamma kv^{-\gamma-1}=\frac{-\gamma k}{v^{\gamma+1}}$$

$$\frac{dp}{dt}=\frac{-\gamma k}{v^{\gamma+1}}\times\frac{dv}{dt}$$

Since $k=pv^{\gamma}$,

$$\frac{dp}{dt}=\frac{-\gamma(pv^{\gamma})}{v^{\gamma+1}}\frac{dv}{dt}=\frac{-\gamma pv^{\gamma}}{v^{\gamma}v^{1}}\frac{dv}{dt}$$

i.e. $$\boldsymbol{\frac{dp}{dt}=-\gamma\frac{p}{v}\frac{dv}{dt}}$$