

Module 4

Parametric Equations

Parametric Equations

Certain mathematical functions can be expressed more simply by expressing, say, x and y separately in terms of a third variable. For example, $y = r \sin \theta$, $x = r \cos \theta$. Then, any value given to θ will produce a pair of values for x and y , which may be plotted to provide a curve of $y = f(x)$.

The third variable, θ , is called a **parameter** and the two expressions for y and x are called **parametric equations**.

The above example of $y = r \sin \theta$ and $x = r \cos \theta$ are the parametric equations for a circle. The equation of any point on a circle, centre at the origin and of radius r is given by: $x^2 + y^2 = r^2$

To show that $y = r \sin \theta$ and $x = r \cos \theta$ are suitable parametric equations for such a circle:

$$\begin{aligned}
 &\text{Left hand side of equation} \\
 &= x^2 + y^2 \\
 &= (r \cos \theta)^2 + (r \sin \theta)^2 \\
 &= r^2 \cos^2 \theta + r^2 \sin^2 \theta \\
 &= r^2 (\cos^2 \theta + \sin^2 \theta) \\
 &= r^2 = \text{right hand side} \\
 &\quad (\text{since } \cos^2 \theta + \sin^2 \theta = 1)
 \end{aligned}$$

A. Common parametric equations

The following are some of the most common parametric equations, and Fig. 21 shows typical shapes of these curves.

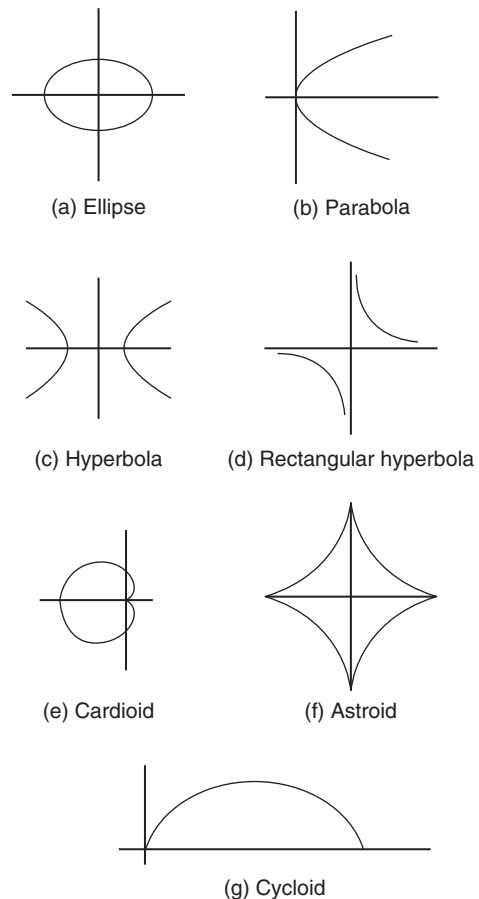


Figure 21

- | | |
|---------------------------|--|
| (a) Ellipse | $x = a \cos \theta, y = b \sin \theta$ |
| (b) Parabola | $x = at^2, y = 2at$ |
| (c) Hyperbola | $x = a \sec \theta, y = b \tan \theta$ |
| (d) Rectangular hyperbola | $x = ct, y = \frac{c}{t}$ |

- (e) Cardioid $x = a(2 \cos \theta - \cos 2\theta)$,
 $y = a(2 \sin \theta - \sin 2\theta)$
 (f) Astroid $x = a \cos^3 \theta$, $y = a \sin^3 \theta$
 (g) Cycloid $x = a(\theta - \sin \theta)$, $y = a(1 - \cos \theta)$

B. Differentiation in parameters

When x and y are given in terms of a parameter say θ , then by the function of a function rule of differentiation

$$\frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx}$$

It may be shown that this can be written as:

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} \quad (1)$$

For the second differential,

$$\frac{d^2y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right) = \frac{d}{d\theta} \left(\frac{dy}{dx} \right) \cdot \frac{d\theta}{dx}$$

or

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{d\theta} \left(\frac{dy}{dx} \right)}{\frac{dx}{d\theta}} \quad (2)$$

Problem 1. Given $x = 5\theta - 1$ and $y = 2\theta(\theta - 1)$, determine $\frac{dy}{dx}$ in terms of θ

$$x = 5\theta - 1, \text{ hence } \frac{dy}{d\theta} = 5$$

$$y = 2\theta(\theta - 1) = 2\theta^2 - 2\theta,$$

$$\text{hence } \frac{dy}{d\theta} = 4\theta - 2 = 2(2\theta - 1)$$

From equation (1),

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{2(2\theta - 1)}{5} \text{ or } \frac{2}{5}(2\theta - 1)$$

Problem 2. The parametric equations of a function are given by $y = 3 \cos 2t$, $x = 2 \sin t$.

Determine expressions for (a) $\frac{dy}{dx}$ (b) $\frac{d^2y}{dx^2}$

(a) $y = 3 \cos 2t$, hence $\frac{dy}{dt} = -6 \sin 2t$

$$x = 2 \sin t, \text{ hence } \frac{dx}{dt} = 2 \cos t$$

From equation (1),

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-6 \sin 2t}{2 \cos t} = \frac{-6(2 \sin t \cos t)}{2 \cos t}$$

(Review double angles)

$$\text{i.e. } \frac{dy}{dx} = -6 \sin t$$

(b) From equation (2),

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt} \left(\frac{dy}{dx} \right)}{\frac{dx}{dt}} = \frac{\frac{d}{dt} (-6 \sin t)}{2 \cos t} = \frac{-6 \cos t}{2 \cos t}$$

$$\text{i.e. } \frac{d^2y}{dx^2} = -3$$

Problem 3. The equation of a tangent drawn to a curve at point (x_1, y_1) is given by:

$$y - y_1 = \frac{dy_1}{dx_1} (x - x_1)$$

Determine the equation of the tangent drawn to the parabola $x = 2t^2$, $y = 4t$ at the point t .

$$\text{At point } t, x_1 = 2t^2, \text{ hence } \frac{dx_1}{dt} = 4t$$

$$\text{and } y_1 = 4t, \text{ hence } \frac{dy_1}{dt} = 4$$

From equation (1),

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{4}{4t} = \frac{1}{t}$$

Hence, the equation of the tangent is:

$$y - 4t = \frac{1}{t}(x - 2t^2)$$

Problem 4. The parametric equations of a cycloid are $x = 4(\theta - \sin \theta)$, $y = 4(1 - \cos \theta)$.

Determine (a) $\frac{dy}{dx}$ (b) $\frac{d^2y}{dx^2}$

(a) $x = 4(\theta - \sin \theta)$

hence $\frac{dx}{d\theta} = 4 - 4 \cos \theta = 4(1 - \cos \theta)$

$y = 4(1 - \cos \theta)$, hence $\frac{dy}{d\theta} = 4 \sin \theta$

From equation (1),

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{4 \sin \theta}{4(1 - \cos \theta)} = \frac{\sin \theta}{(1 - \cos \theta)}$$

(b) From equation (2),

$$\begin{aligned} \frac{d^2y}{dx^2} &= \frac{\frac{d}{d\theta} \left(\frac{dy}{dx} \right)}{\frac{dx}{d\theta}} = \frac{\frac{d}{d\theta} \left(\frac{\sin \theta}{1 - \cos \theta} \right)}{4(1 - \cos \theta)} \\ &= \frac{(1 - \cos \theta)(\cos \theta) - (\sin \theta)(\sin \theta)}{4(1 - \cos \theta)^2} \\ &= \frac{\cos \theta - \cos^2 \theta - \sin^2 \theta}{4(1 - \cos \theta)^3} \\ &= \frac{\cos \theta - (\cos^2 \theta + \sin^2 \theta)}{4(1 - \cos \theta)^3} \\ &= \frac{\cos \theta - 1}{4(1 - \cos \theta)^3} \\ &= \frac{-(1 - \cos \theta)}{4(1 - \cos \theta)^3} = \frac{-1}{4(1 - \cos \theta)^2} \end{aligned}$$

Exercise 17. Parametric Equations

Solved Problems:

Problem 5. The equation of the normal drawn to a curve at point (x_1, y_1) is given by:

$$y - y_1 = -\frac{1}{\frac{dy_1}{dx_1}}(x - x_1)$$

Determine the equation of the normal drawn to the astroid $x = 2 \cos^3 \theta$, $y = 2 \sin^3 \theta$ at the point $\theta = \frac{\pi}{4}$

$$x = 2 \cos^3 \theta, \text{ hence } \frac{dx}{d\theta} = -6 \cos^2 \theta \sin \theta$$

$$y = 2 \sin^3 \theta, \text{ hence } \frac{dy}{d\theta} = 6 \sin^2 \theta \cos \theta$$

From equation (1),

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{6 \sin^2 \theta \cos \theta}{-6 \cos^2 \theta \sin \theta} = -\frac{\sin \theta}{\cos \theta} = -\tan \theta$$

$$\text{When } \theta = \frac{\pi}{4}, \quad \frac{dy}{dx} = -\tan \frac{\pi}{4} = -1$$

$$x_1 = 2 \cos^3 \frac{\pi}{4} = 0.7071 \text{ and } y_1 = 2 \sin^3 \frac{\pi}{4} = 0.7071$$

Hence, **the equation of the normal is:**

$$y - 0.7071 = -\frac{1}{-1}(x - 0.7071)$$

i.e. $y - 0.7071 = x - 0.7071$

i.e. $y = x$

Problem 6. The parametric equations for a hyperbola are $x = 2 \sec \theta$, $y = 4 \tan \theta$. Evaluate

(a) $\frac{dy}{dx}$ (b) $\frac{d^2y}{dx^2}$, correct to 4 significant figures, when $\theta = 1$ radian

(a) $x = 2 \sec \theta$, hence $\frac{dx}{d\theta} = 2 \sec \theta \tan \theta$

$y = 4 \tan \theta$, hence $\frac{dy}{d\theta} = 4 \sec^2 \theta$

From equation (1),

$$\begin{aligned} \frac{dy}{dx} &= \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{4 \sec^2 \theta}{2 \sec \theta \tan \theta} = \frac{2 \sec \theta}{\tan \theta} \\ &= \frac{2 \left(\frac{1}{\cos \theta} \right)}{\left(\frac{\sin \theta}{\cos \theta} \right)} = \frac{2}{\sin \theta} \text{ or } 2 \operatorname{cosec} \theta \end{aligned}$$

When $\theta = 1$ rad, $\frac{dy}{dx} = \frac{2}{\sin 1} = 2.377$, correct to 4 significant figures.

(b) From equation (2),

$$\begin{aligned}\frac{d^2y}{dx^2} &= \frac{\frac{d}{d\theta} \left(\frac{dy}{dx} \right)}{\frac{dx}{d\theta}} = \frac{\frac{d}{d\theta} (2 \operatorname{cosec} \theta)}{2 \sec \theta \tan \theta} \\ &= \frac{-2 \operatorname{cosec} \theta \cot \theta}{2 \sec \theta \tan \theta} \\ &= -\left(\frac{1}{\sin \theta} \right) \left(\frac{\cos \theta}{\sin \theta} \right) \\ &= \frac{\left(\frac{1}{\cos \theta} \right) \left(\frac{\sin \theta}{\cos \theta} \right)}{\left(\frac{1}{\cos \theta} \right) \left(\frac{\sin \theta}{\cos \theta} \right)} \\ &= -\left(\frac{\cos \theta}{\sin^2 \theta} \right) \left(\frac{\cos^2 \theta}{\sin \theta} \right) \\ &= -\frac{\cos^3 \theta}{\sin^3 \theta} = -\cot^3 \theta\end{aligned}$$

When $\theta = 1$ rad, $\frac{d^2y}{dx^2} = -\cot^3 1 = -\frac{1}{(\tan 1)^3} = -0.2647$, correct to 4 significant figures.

$$\begin{aligned}\text{Hence, radius of curvature, } \rho &= \frac{\sqrt{\left[1 + \left(\frac{dy}{dx} \right)^2 \right]^3}}{\frac{d^2y}{dx^2}} \\ &= \frac{\sqrt{\left[1 + \left(\frac{1}{t} \right)^2 \right]^3}}{-\frac{1}{6t^3}}\end{aligned}$$

$$\begin{aligned}\text{When } t = 2, \rho &= \frac{\sqrt{\left[1 + \left(\frac{1}{2} \right)^2 \right]^3}}{-\frac{1}{6(2)^3}} = \frac{\sqrt{(1.25)^3}}{-\frac{1}{48}} \\ &= -48\sqrt{(1.25)^3} = -67.08.\end{aligned}$$

Problem 7. When determining the surface tension of a liquid, the radius of curvature, ρ , of part of the surface is given by:

$$\rho = \frac{\sqrt{\left[1 + \left(\frac{dy}{dx} \right)^2 \right]^3}}{\frac{d^2y}{dx^2}}$$

Find the radius of curvature of the part of the surface having the parametric equations $x = 3t^2$, $y = 6t$ at the point $t = 2$

$$x = 3t^2, \text{ hence } \frac{dx}{dt} = 6t$$

$$y = 6t, \text{ hence } \frac{dy}{dt} = 6$$

$$\text{From equation (1), } \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{6}{6t} = \frac{1}{t}$$

From equation (2),

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt} \left(\frac{dy}{dx} \right)}{\frac{dx}{dt}} = \frac{\frac{d}{dt} \left(\frac{1}{t} \right)}{6t} = \frac{-\frac{1}{t^2}}{6t} = -\frac{1}{6t^3}$$

Exercise 18. Differentiation of Parametric Equations