Module 3 - Parametric Equations

A. Introduction to parametric equations

Certain mathematical functions can be expressed more simply by expressing, say, x and y separately in terms of a third variable. For example, $y = r \sin\theta$, $x = r\cos\theta$. Then, any value given to θ will produce a pair of values for x and y, which may be plotted to provide a curve of $y = f(x)$.

The third variable, θ, is called a **parameter** and the two expressions for y and x are called **parametric equations**.

The above example of $y = r\sin\theta$ and $x = r\cos\theta$ are the parametric equations for a circle. The equation of any point on a circle, centre at the origin and of radius r is given by: $x^2 + y^2 = r^2$.

To show that $y = r\sin\theta$ and $x = r\cos\theta$ are suitable parametric equations for such a circle:

Left hand side of equation

$$\begin{aligned} &= x^2 + y^2 \\ &= (r\cos\theta)^2 + (r\sin\theta)^2 \\ &= r^2\cos^2\theta + r^2\sin^2\theta \\ &= r^2\left(\cos^2\theta + \sin^2\theta\right) \\ &= r^2 = \text{right hand side} \\ &\quad (\text{since } \cos^2\theta + \sin^2\theta = 1) \end{aligned}$$

B. Common Parametric Equations

The following are some of the most common param-etric equations, and Figure 18 shows typical shapes of these curves.

(a) Ellipse $x = a\cos\theta,\ y = b\sin\theta$

(b) Parabola $x = a\,t^2,\ y = 2a\,t$

(c) Hyperbola $x = a\sec\theta,\ y = b\tan\theta$

(d) Rectangular hyperbola $x = c\,t,\ y = \dfrac{c}{t}$

(e) Cardioid $x = a\,(2\cos\theta - \cos 2\theta)$, $y = a\,(2\sin\theta - \sin 2\theta)$

(f) Astroid $x = a\cos^3\theta,\ y = a\sin^3\theta$

(g) Cycloid $x = a\,(\theta - \sin\theta),\ y = a\,(1 - \cos\theta)$

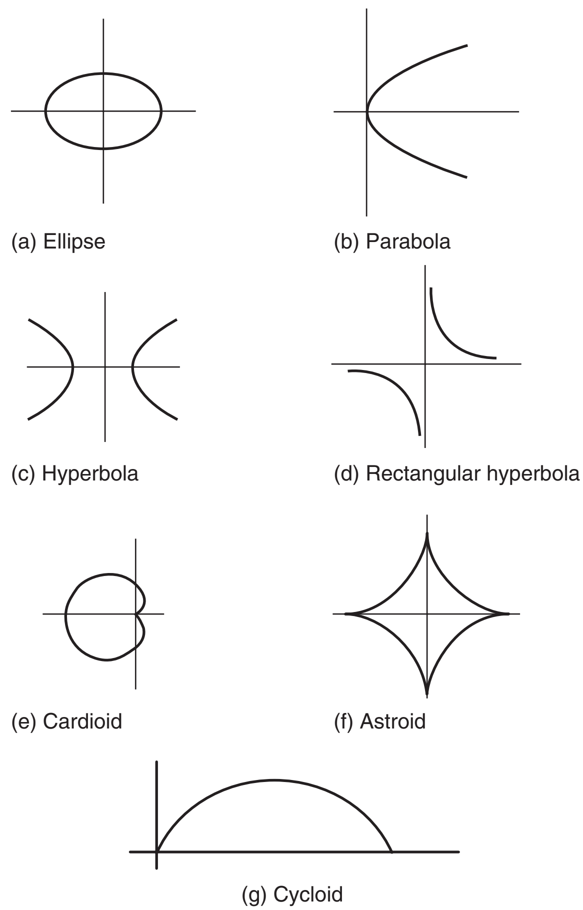

Figure 18

C. Differentiation in Parameters

When x and y are given in terms of a parameter, say θ, then by the function of a function rule of differentiation.

$$\frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx}$$

It may be shown that this can be written as:

$$\boxed{\frac{\mathbf{dy}}{\mathbf{dx}} = \frac{\dfrac{\mathbf{dy}}{\mathbf{d\theta}}}{\dfrac{\mathbf{dx}}{\mathbf{d\theta}}}} \qquad (1)$$

For the second differential,

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{d\theta}\left(\frac{dy}{dx}\right) \cdot \frac{d\theta}{dx}$$

or

$$\boxed{\frac{\mathbf{d^2y}}{\mathbf{dx^2}} = \frac{\dfrac{\mathbf{d}}{\mathbf{d\theta}}\left(\dfrac{\mathbf{dy}}{\mathbf{dx}}\right)}{\dfrac{\mathbf{dx}}{\mathbf{d\theta}}}} \qquad (2)$$

Problem 1. Given $x = 5\theta - 1$ and $y = 2\theta\,(\theta - 1)$, determine $\frac{dy}{dx}$ in terms of θ

$x = 5\theta - 1$, hence $\frac{dy}{d\theta} = 5$

$y = 2\theta(\theta - 1) = 2\theta^2 - 2\theta$,

hence $\frac{dy}{d\theta} = 4\theta - 2 = 2\,(2\theta - 1)$

From equation (1),

$$\frac{dy}{dx} = \frac{\dfrac{dy}{d\theta}}{\dfrac{dx}{d\theta}} = \frac{\mathbf{2(2\theta - 1)}}{\mathbf{5}} \text{ or } \frac{\mathbf{2}}{\mathbf{5}}(\mathbf{2\theta - 1})$$

Problem 2. The parametric equations of a function are given by $y = 3\cos 2t$, $x = 2\sin t$. Determine expressions for (a) $\frac{dy}{dx}$ (b) $\frac{d^2y}{dx^2}$

(a) $y = 3\cos 2t$, hence $\frac{dy}{dt} = -6\sin 2t$

$x = 2\sin t$, hence $\frac{dx}{dt} = 2\cos t$

From equation (1),

$$\frac{dy}{dx} = \frac{\dfrac{dy}{dt}}{\dfrac{dx}{dt}} = \frac{-6\sin 2t}{2\cos t} = \frac{-6(2\sin t\cos t)}{2\cos t}$$

from double angles.

i.e. $\frac{\mathbf{dy}}{\mathbf{dx}} = \mathbf{-6\sin t}$

(b) From equation (2),

$$\frac{d^2y}{dx^2} = \frac{\dfrac{d}{dt}\left(\dfrac{dy}{dx}\right)}{\dfrac{dx}{dt}} = \frac{\dfrac{d}{dt}(-6\sin t)}{2\cos t} = \frac{-6\cos t}{2\cos t}$$

i.e. $\frac{\mathbf{d^2y}}{\mathbf{dx^2}} = \mathbf{-3}$

Problem 3. The equation of a tangent drawn to a curve at point (x_1, y_1) is given by:

$$y - y_1 = \frac{dy_1}{dx_1}(x - x_1)$$

Determine the equation of the tangent drawn to the parabola $x = 2t^2$, $y = 4t$ at the point t.

At point t, $x_1 = 2t^2$, hence $\frac{dx_1}{dt} = 4t$

and $y_1 = 4t$, hence $\frac{dy_1}{dt} = 4$

From equation (1),

$$\frac{dy}{dx} = \frac{\dfrac{dy}{dt}}{\dfrac{dx}{dt}} = \frac{4}{4t} = \frac{1}{t}$$

Hence, the equation of the tangent is: $\boldsymbol{y - 4t = \frac{1}{t}\left(x - 2t^2\right)}$

Problem 4. The parametric equations of a cycloid are $x = 4(\theta - \sin\theta)$, $y = 4(1 - \cos\theta)$. Determine (a) $\frac{dy}{dx}$ (b) $\frac{d^2y}{dx^2}$

(a) $x = 4(\theta - \sin\theta)$,

hence $\dfrac{dx}{d\theta} = 4 - 4\cos\theta = 4(1 - \cos\theta)$

$y = 4(1 - \cos\theta)$, hence $\dfrac{dy}{d\theta} = 4\sin\theta$

From equation (1),

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{4\sin\theta}{4(1-\cos\theta)} = \boldsymbol{\frac{\sin\theta}{(1-\cos\theta)}}$$

(b) From equation (2),

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{d\theta}\left(\frac{dy}{dx}\right)}{\frac{dx}{d\theta}} = \frac{\frac{d}{d\theta}\left(\frac{\sin\theta}{1-\cos\theta}\right)}{4(1-\cos\theta)}$$

$$= \frac{\frac{(1-\cos\theta)(\cos\theta) - (\sin\theta)(\sin\theta)}{(1-\cos\theta)^2}}{4(1-\cos\theta)}$$

$$= \frac{\cos\theta - \cos^2\theta - \sin^2\theta}{4(1-\cos\theta)^3}$$

$$= \frac{\cos\theta - \left(\cos^2\theta + \sin^2\theta\right)}{4(1-\cos\theta)^3}$$

$$= \frac{\cos\theta - 1}{4(1-\cos\theta)^3}$$

$$= \frac{-(1-\cos\theta)}{4(1-\cos\theta)^3} = \boldsymbol{\frac{-1}{4(1-\cos\theta)^2}}$$

Exercise 12. Differentiation of parametric equations

Solved problems on differentiation of parametric equations

Problem 5. The equation of the normal drawn to a curve at point (x_1, y_1) is given by:

$$y - y_1 = -\frac{1}{\frac{dy_1}{dx_1}}(x - x_1)$$

Determine the equation of the normal drawn to the astroid $x = 2\cos^3\theta$, $y = 2\sin^3\theta$ at the point $\theta = \frac{\pi}{4}$

$$x = 2\cos^3\theta, \text{ hence } \frac{dx}{d\theta} = -6\cos^2\theta \sin\theta$$

$$y = 2\sin^3\theta, \text{ hence } \frac{dy}{d\theta} = 6\sin^2\theta \cos\theta$$

From equation (1),

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{6\sin^2\theta\cos\theta}{-6\cos^2\theta\sin\theta} = -\frac{\sin\theta}{\cos\theta} = -\tan\theta$$

When $\theta = \frac{\pi}{4}$, $\frac{dy}{dx} = -\tan\frac{\pi}{4} = -1$

$x_1 = 2\cos^3\frac{\pi}{4} = 0.7071$ and $y_1 = 2\sin^3\frac{\pi}{4} = 0.7071$

Hence, **the equation of the normal is**:

$$y - 0.7071 = -\frac{1}{-1}(x - 0.7071)$$

i.e. $y - 0.7071 = x - 0.7071$

i.e. $\boldsymbol{y = x}$

Problem 6. The parametric equations for a hyperbola are $x = 2\sec\theta$, $y = 4\tan\theta$. Evaluate (a) $\frac{dy}{dx}$ (b) $\frac{d^2y}{dx^2}$, correct to 4 significant figures, when $\theta = 1$ radian.

(a) $x = 2\sec\theta$, hence $\frac{dx}{d\theta} = 2\sec\theta\tan\theta$

$y = 4\tan\theta$, hence $\frac{dy}{d\theta} = 4\sec^2\theta$

From equation (1),

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{4\sec^2\theta}{2\sec\theta\tan\theta} = \frac{2\sec\theta}{\tan\theta}$$

$$= \frac{2\left(\frac{1}{\cos\theta}\right)}{\left(\frac{\sin\theta}{\cos\theta}\right)} = \frac{2}{\sin\theta} \text{ or } 2\operatorname{cosec}\theta$$

When $\theta = 1$ rad, $\frac{dy}{dx} = \frac{2}{\sin 1} = \mathbf{2.377}$, correct to 4 significant figures.

(b) From equation (2),

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{d\theta}\left(\frac{dy}{dx}\right)}{\frac{dx}{d\theta}} = \frac{\frac{d}{d\theta}(2\operatorname{cosec}\theta)}{2\sec\theta\tan\theta}$$

$$= \frac{-2\operatorname{cosec}\theta\cot\theta}{2\sec\theta\tan\theta}$$

$$= \frac{-\left(\frac{1}{\sin\theta}\right)\left(\frac{\cos\theta}{\sin\theta}\right)}{\left(\frac{1}{\cos\theta}\right)\left(\frac{\sin\theta}{\cos\theta}\right)}$$

$$= -\left(\frac{\cos\theta}{\sin^2\theta}\right)\left(\frac{\cos^2\theta}{\sin\theta}\right)$$

$$= -\frac{\cos^3\theta}{\sin^3\theta} = -\cot^3\theta$$

When $\theta = 1$ rad, $\frac{d^2y}{dx^2} = -\cot^3 1 = -\frac{1}{(\tan 1)^3}$ $= \mathbf{-0.2647}$, correct to 4 significant figures.

Problem 7. When determining the surface tension of a liquid, the radius of curvature, ρ, of part of the surface is given by:

$$\rho = \frac{\sqrt{\left[1+\left(\frac{dy}{dx}\right)^2\right]^3}}{\frac{d^2y}{dx^2}}$$

Find the radius of curvature of the part of the surface having the parametric equations $x = 3t^2$, $y = 6t$ at the point $t = 2$.

$x = 3t^2$, hence $\frac{dx}{dt} = 6t$

$y = 6t$, hence $\frac{dy}{dt} = 6$

From equation (1), $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{6}{6t} = \frac{1}{t}$

From equation (2),

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{\frac{d}{dt}\left(\frac{1}{t}\right)}{6t} = \frac{-\frac{1}{t^2}}{6t} = -\frac{1}{6t^3}$$

Hence, radius of curvature, $\rho = \frac{\sqrt{\left[1+\left(\frac{dy}{dx}\right)^2\right]^3}}{\frac{d^2y}{dx^2}}$

$$= \frac{\sqrt{\left[1+\left(\frac{1}{t}\right)^2\right]^3}}{-\frac{1}{6t^3}}$$

When $t = 2$, $\rho = \frac{\sqrt{\left[1+\left(\frac{1}{2}\right)^2\right]^3}}{-\frac{1}{6(2)^3}} = \frac{\sqrt{(1.25)^3}}{-\frac{1}{48}}$

$$= -48\sqrt{(1.25)^3} = \mathbf{-67.08}$$

Exercise 13. Differentiation of parametric equations