Module 2 - Applications of Differentiation

A. Rates of Change

If a quantity y depends on and varies with a quantity x then the rate of change of y with respect to x is $\frac{dy}{dx}$. Thus, for example, the rate of change of pressure p with height h is $\frac{dp}{dh}$.

A rate of change with respect to time is usually just called 'the rate of change', the 'with respect to time' being assumed. Thus, for example, a rate of change of current, i, is $\frac{di}{dt}$ and a rate of change of temperature, θ, is $\frac{d\theta}{dt}$, and so on.

Problem 1. The length l metres of a certain metal rod at temperature θ°C is given by $l = 1 + 0.00005\theta + 0.0000004\theta^2$. Determine the rate of change of length, in mm/°C, when the temperature is (a) 100°C and (b) 400°C.

The rate of change of length means $\frac{dl}{d\theta}$.

Since length $\quad l = 1 + 0.00005\theta + 0.0000004\theta^2$,

then $\quad \frac{dl}{d\theta} = 0.00005 + 0.0000008\theta$

(a) When $\theta = 100$°C,

$$\frac{dl}{d\theta} = 0.00005 + (0.0000008)(100)$$
$$= 0.00013\,\text{m/°C}$$
$$= \mathbf{0.13\,mm/°C}$$

(b) When $\theta = 400$°C,

$$\frac{dl}{d\theta} = 0.00005 + (0.0000008)(400)$$
$$= 0.00037\,\text{m/°C}$$
$$= \mathbf{0.37\,mm/°C}$$

Problem 2. The luminous intensity I candelas of a lamp at varying voltage V is given by $I = 4 \times 10^{-4}\,V^2$. Determine the voltage at which the light is increasing at a rate of 0.6 candelas per volt.

The rate of change of light with respect to voltage is given by $\frac{dI}{dV}$.

Since $\quad I = 4 \times 10^{-4}\,V^2$,

$$\frac{dI}{dV} = (4 \times 10^{-4})(2)V = 8 \times 10^{-4}\,V$$

When the light is increasing at 0.6 candelas per volt then $+0.6 = 8 \times 10^{-4}\,V$, from which, voltage

$$V = \frac{0.6}{8 \times 10^{-4}} = 0.075 \times 10^{+4}$$
$$= \mathbf{750\ volts}$$

Problem 3. Newtons law of cooling is given by $\theta = \theta_0 e^{-kt}$, where the excess of temperature at zero time is θ_0°C and at time t seconds is θ°C. Determine the rate of change of temperature after 40 s, given that $\theta_0 = 16$°C and $k = -0.03$.

The rate of change of temperature is $\frac{d\theta}{dt}$.

Since $\quad \theta = \theta_0 e^{-kt}$

then $\quad \frac{d\theta}{dt} = (\theta_0)(-k)e^{-kt} = -k\theta_0 e^{-kt}$

When $\quad \theta_0 = 16, k = -0.03 \quad$ and $\quad t = 40$

then $\quad \frac{d\theta}{dt} = -(-0.03)(16)e^{-(-0.03)(40)}$

$$= 0.48e^{1.2} = \mathbf{1.594°C/s}$$

Problem 4. The displacement s cm of the end of a stiff spring at time t seconds is given by $s = a\mathrm{e}^{-kt} \sin 2\pi ft$. Determine the velocity of the end of the spring after 1 s, if $a = 2$, $k = 0.9$ and $f = 5$.

Velocity, $v = \frac{\mathrm{d}s}{\mathrm{d}t}$ where $s = a\mathrm{e}^{-kt} \sin 2\pi ft$ (i.e. a product).

Using the product rule,

$$\frac{\mathrm{d}s}{\mathrm{d}t} = (a\mathrm{e}^{-kt})(2\pi f \cos 2\pi ft) + (\sin 2\pi ft)(-ak\mathrm{e}^{-kt})$$

When $a = 2$, $k = 0.9$, $f = 5$ and $t = 1$,

$$\textbf{velocity}, \boldsymbol{v} = (2\mathrm{e}^{-0.9})(2\pi 5 \cos 2\pi 5) + (\sin 2\pi 5)(-2)(0.9)\mathrm{e}^{-0.9}$$
$$= 25.5455 \cos 10\pi - 0.7318 \sin 10\pi$$
$$= 25.5455(1) - 0.7318(0)$$
$$= \mathbf{25.55\ cm/s}$$

(Note that $\cos 10\pi$ means 'the cosine of 10π radians', *not* degrees, and $\cos 10\pi \equiv \cos 2\pi = 1$).

Exercise 6. Rates of change

B. Velocity and Acceleration

When a car moves a distance x metres in a time t seconds along a straight road, if the velocity v is constant then $v = \frac{x}{t}$ m/s, i.e. the gradient of the distance/time graph shown in Fig. 6 is constant.

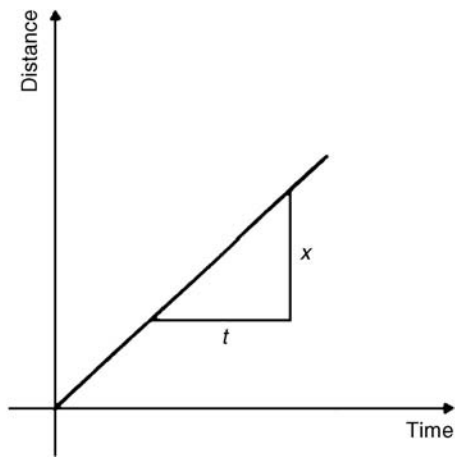

Figure 6

If, however, the velocity of the car is not constant then the distance/time graph will not be a straight line. It may be as shown in Fig. 7.

The average velocity over a small time δt and distance δx is given by the gradient of the chord AB, i.e. the average velocity over time δt is $\frac{\delta x}{\delta t}$.

As $\delta t \to 0$, the chord AB becomes a tangent, such that at point A, the velocity is given by:

$$v = \frac{\mathrm{d}x}{\mathrm{d}t}$$

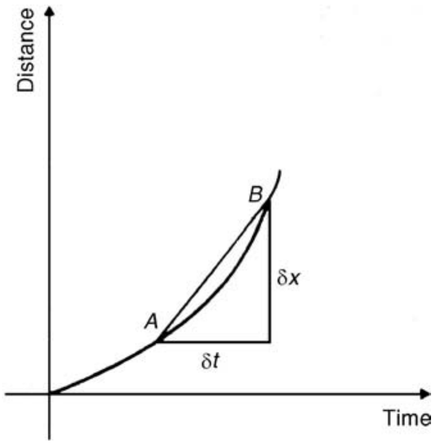

Figure 7

Hence the velocity of the car at any instant is given by the gradient of the distance/time graph. If an expression for the distance x is known in terms of time t then the velocity is obtained by differentiating the expression.

The acceleration a of the car is defined as the rate of change of velocity. A velocity/time graph is shown in Fig. 8. If δv is the change in v and δt the corresponding change in time, then $a = \dfrac{\delta v}{\delta t}$.

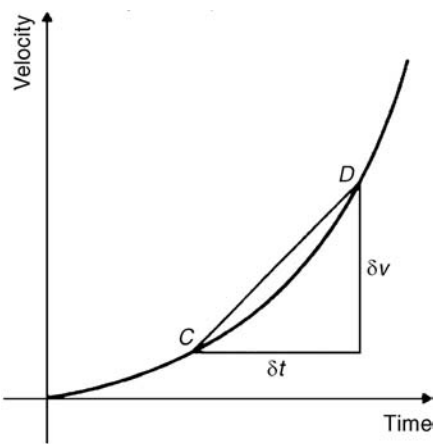

Figure 8

As $\delta t \to 0$, the chord CD becomes a tangent, such that at point C, the acceleration is given by:

$$a = \frac{dv}{dt}$$

Hence the acceleration of the car at any instant is given by the gradient of the velocity/time graph. If an expression for velocity is known in terms of time t then the acceleration is obtained by differentiating the expression.

Acceleration $a = \dfrac{dv}{dt}$. However, $v = \dfrac{dx}{dt}$. Hence $a = \dfrac{d}{dt}\left(\dfrac{dx}{dt}\right) = \dfrac{d^2x}{dx^2}$

The acceleration is given by the second differential coefficient of distance x with respect to time t.

Summarising, if a body moves a distance x metres in a time t seconds then:

(i) **distance $x = f(t)$.**

(ii) **velocity $v = f'(t)$ or $\dfrac{dx}{dt}$**, which is the gradient of the distance/time graph.

(iii) **acceleration $a = \dfrac{dv}{dt} = f''(t)$ or $\dfrac{d^2x}{dt^2}$**, which is the gradient of the velocity/time graph.

Problem 5. The distance x metres moved by a car in a time t seconds is given by $x = 3t^3 - 2t^2 + 4t - 1$. Determine the velocity and acceleration when (a) $t = 0$ and (b) $t = 1.5$ s.

Distance $\quad x = 3t^3 - 2t^2 + 4t - 1$ m

Velocity $\quad v = \dfrac{dx}{dt} = 9t^2 - 4t + 4$ m/s

Acceleration $\quad a = \dfrac{d^2x}{dx^2} = 18t - 4$ m/s^2

(a) When time $t = 0$,
velocity $v = 9(0)^2 - 4(0) + 4 = $ **4 m/s** and
acceleration $a = 18(0) - 4 = $ **−4 m/s²** (i.e. a deceleration)

(b) When time $t = 1.5$ s,
velocity $v = 9(1.5)^2 - 4(1.5) + 4 = $ **18.25 m/s**
and acceleration $a = 18(1.5) - 4 = $ **23 m/s²**

Problem 6. Supplies are dropped from a helicoptor and the distance fallen in a time t seconds is given by $x = \frac{1}{2}gt^2$, where $g = 9.8$ m/s^2. Determine the velocity and acceleration of the supplies after it has fallen for 2 seconds.

Distance $x = \frac{1}{2}gt^2 = \frac{1}{2}(9.8)t^2 = 4.9t^2$ m

Velocity $v = \frac{dv}{dt} = 9.8t$ m/s

and acceleration $a = \frac{d^2x}{dt^2} = 9.8$ m/s^2

When time $t = 2$ s,

$$\textbf{velocity}, \ v = (9.8)(2) = \mathbf{19.6\,m/s}$$

and **acceleration** $\boldsymbol{a = 9.8\,m/s^2}$

(which is acceleration due to gravity).

Problem 7. The distance x metres travelled by a vehicle in time t seconds after the brakes are applied is given by $x = 20t - \frac{5}{3}t^2$. Determine (a) the speed of the vehicle (in km/h) at the instant the brakes are applied, and (b) the distance the car travels before it stops.

(a) Distance, $x = 20t - \frac{5}{3}t^2$.

Hence velocity $v = \frac{dx}{dt} = 20 - \frac{10}{3}t$.

At the instant the brakes are applied, time = 0.

Hence **velocity**, $v = 20$ m/s

$$= \frac{20 \times 60 \times 60}{1000} \text{ km/h}$$

$$= \mathbf{72\,km/h}$$

(Note: changing from m/s to km/h merely involves multiplying by 3.6).

(b) When the car finally stops, the velocity is zero, i.e. $v = 20 - \frac{10}{3}t = 0$, from which, $20 = \frac{10}{3}t$, giving $t = 6$ s.

Hence the distance travelled before the car stops is given by:

$$x = 20t - \tfrac{5}{3}t^2 = 20(6) - \tfrac{5}{3}(6)^2$$

$$= 120 - 60 = \mathbf{60\,m}$$

Problem 8. The angular displacement θ radians of a flywheel varies with time t seconds and follows the equation $\theta = 9t^2 - 2t^3$. Determine (a) the angular velocity and acceleration of the flywheel when time, $t = 1$ s, and (b) the time when the angular acceleration is zero.

(a) Angular displacement $\theta = 9t^2 - 2t^3$ rad

Angular velocity $\omega = \frac{d\theta}{dt} = 18t - 6t^2$ rad/s

When time $t = 1$ s,

$$\omega = 18(1) - 6(1)^2 = \mathbf{12\,rad/s}$$

Angular acceleration $\alpha = \frac{d^2\theta}{dt^2} = 18 - 12t$ rad/s^2

When time $t = 1$ s,

$$\alpha = 18 - 12(1) = \mathbf{6\,rad/s^2}$$

(b) When the angular acceleration is zero, $18 - 12t = 0$, from which, $18 = 12t$, giving time, $\boldsymbol{t = 1.5}$ **s**.

Problem 9. The displacement x cm of the slide valve of an engine is given by $x = 2.2\cos 5\pi t + 3.6\sin 5\pi t$. Evaluate the velocity (in m/s) when time $t = 30$ ms.

Displacement $x = 2.2\cos 5\pi t + 3.6\sin 5\pi t$

Velocity $v = \frac{dx}{dt}$

$$= (2.2)(-5\pi)\sin 5\pi t + (3.6)(5\pi)\cos 5\pi t$$

$$= -11\pi \sin 5\pi t + 18\pi \cos 5\pi t \text{ cm/s}$$

When time $t = 30$ ms, velocity

$$= -11\pi \sin\left(5\pi \cdot \frac{30}{10^3}\right) + 18\pi \cos\left(5\pi \cdot \frac{30}{10^3}\right)$$

$$= -11\pi \sin 0.4712 + 18\pi \cos 0.4712$$

$$= -11\pi \sin 27^\circ + 18\pi \cos 27^\circ$$

$$= -15.69 + 50.39 = 34.7 \text{ cm/s}$$

$$= \mathbf{0.347\,m/s}$$

Exercise 7. Velocity and acceleration

C. Turning points

In Fig. 9, the gradient (or rate of change) of the curve changes from positive between O and P to negative between P and Q, and then positive again between Q and R. At point P, the gradient is zero and, as x increases, the gradient of the curve changes from positive just before P to negative just after. Such a point is called a **maximum point** and appears as the 'crest of a wave'. At point Q, the gradient is also zero and, as x increases, the gradient of the curve changes from negative just before Q to positive just after. Such a point is called a **minimum point**, and appears as the 'bottom of a valley'. Points such as P and Q are given the general name of **turning points**.

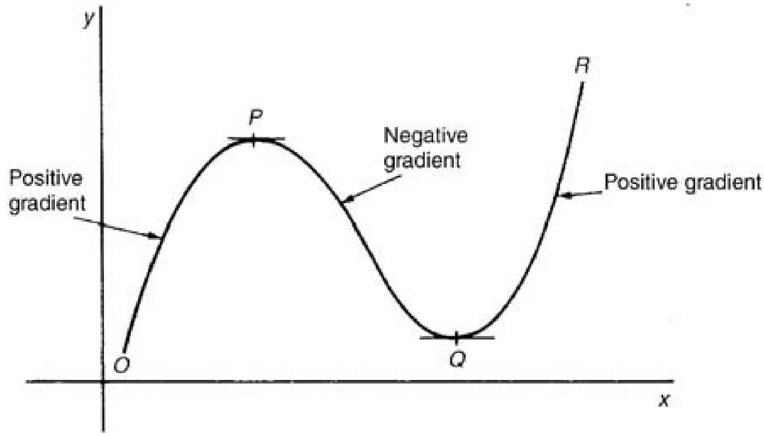

Figure 9

It is possible to have a turning point, the gradient on either side of which is the same. Such a point is given the special name of a **point of inflexion**, and examples are shown in Fig. 10.

Maximum and minimum points and points of inflexion are given the general term of **stationary points**.

Procedure for finding and distinguishing between stationary points:

(i) Given $y = f(x)$, determine $\frac{dy}{dx}$ (i.e. $f'(x)$)

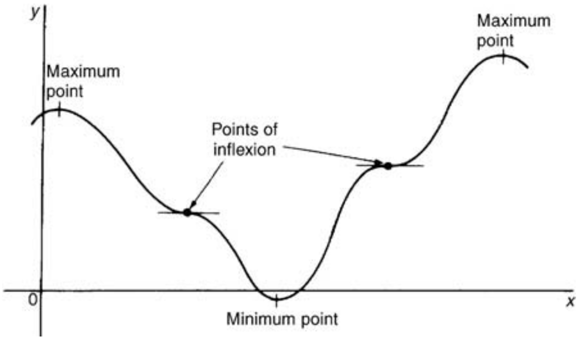

Figure 10

(ii) Let $\frac{dy}{dx} = 0$ and solve for the values of x.

(iii) Substitute the values of x into the original equation, $y = f(x)$, to find the corresponding y-ordinate values. This establishes the co-ordinates of the stationary points.

To determine the nature of the stationary points: Either

(iv) Find $\frac{d^2y}{dx^2}$ and substitute into it the values of x found in (ii).
If the result is:
(a) positive—the point is a minimum one,
(b) negative—the point is a maximum one,
(c) zero—the point is a point of inflexion

or

(v) Determine the sign of the gradient of the curve just before and just after the stationary points. If the sign change for the gradient of the curve is:
(a) positive to negative—the point is a maximum one
(b) negative to positive—the point is a minimum one
(c) positive to positive or negative to negative—the point is a point of inflexion

Problem 10. Locate the turning point on the curve $y = 3x^2 - 6x$ and determine its nature by examining the sign of the gradient on either side.

Following the above procedure:

(i) Since $y = 3x^2 - 6x$, $\frac{dy}{dx} = 6x - 6$.

(ii) At a turning point, $\frac{dy}{dx} = 0$. Hence $6x - 6 = 0$, from which, $x = 1$.

(iii) When $x = 1$, $y = 3(1)^2 - 6(1) = -3$.

Hence the co-ordinates of the turning point are (1, −3).

(iv) If x is slightly less than 1, say, 0.9, then

$$\frac{dy}{dx} = 6(0.9) - 6 = -0.6,$$

i.e. negative.
If x is slightly greater than 1, say, 1.1, then

$$\frac{dy}{dx} = 6(1.1) - 6 = 0.6,$$

i.e. positive.

Since the gradient of the curve is negative just before the turning point and positive just after (i.e. $- \vee +$), **(1, −3) is a minimum point**.

Problem 11. Find the maximum and minimum values of the curve $y = x^3 - 3x + 5$ by

(a) examining the gradient on either side of the turning points, and
(b) determining the sign of the second derivative.

Since $y = x^3 - 3x + 5$ then $\frac{dy}{dx} = 3x^2 - 3$

For a maximum or minimum value $\frac{dy}{dx} = 0$

Hence $3x^2 - 3 = 0$, from which, $3x^2 = 3$ and $x = \pm 1$

When $x = 1$, $y = (1)^3 - 3(1) + 5 = 3$

When $x = -1$, $y = (-1)^3 - 3(-1) + 5 = 7$

Hence (1, 3) and (−1, 7) are the co-ordinates of the turning points.

(a) Considering the point (1, 3):
If x is slightly less than 1, say 0.9, then

$$\frac{dy}{dx} = 3(0.9)^2 - 3,$$

which is negative.
If x is slightly more than 1, say 1.1, then

$$\frac{dy}{dx} = 3(1.1)^2 - 3,$$

which is positive.

Since the gradient changes from negative to positive, **the point (1, 3) is a minimum point**.

Considering the point $(-1, 7)$:

If x is slightly less than -1, say -1.1, then

$$\frac{dy}{dx} = 3(-1.1)^2 - 3,$$

which is positive.

If x is slightly more than -1, say -0.9, then

$$\frac{dy}{dx} = 3(-0.9)^2 - 3,$$

which is negative.

Since the gradient changes from positive to negative, **the point (−1, 7) is a maximum point**.

(b) Since $\frac{dy}{dx} = 3x^2 - 3$, then $\frac{d^2y}{dx^2} = 6x$

When $x = 1$, $\frac{d^2y}{dx^2}$ is positive, hence (1, 3) is a **minimum value**.

When $x = -1$, $\frac{d^2y}{dx^2}$ is negative, hence $(-1, 7)$ is a **maximum value**.

Thus the maximum value is 7 and the minimum value is 3.

It can be seen that the second differential method of determining the nature of the turning points is, in this case, quicker than investigating the gradient.

Problem 12. Locate the turning point on the following curve and determine whether it is a maximum or minimum point: $y = 4\theta + e^{-\theta}$.

Since $\quad y = 4\theta + e^{-\theta}$

then $\quad \frac{dy}{d\theta} = 4 - e^{-\theta} = 0$

for a maximum or minimum value.

Hence $4 = e^{-\theta}$, $\frac{1}{4} = e^{\theta}$, giving $\theta = \ln \frac{1}{4} = -1.3863$

$$\text{When } \theta = -1.3863,\ y = 4(-1.3863) + e^{-(-1.3863)} = 5.5452 + 4.0000 = -1.5452.$$

Thus $(-1.3863, -1.5452)$ are the co-ordinates of the turning point.

$$\frac{d^2y}{d\theta^2} = e^{-\theta}.$$

When $\theta = -1.3863$,

$$\frac{d^2y}{d\theta^2} = e^{+1.3863} = 4.0,$$

which is positive, hence **(−1.3863, −1.5452) is a minimum point**.

Problem 13. Determine the co-ordinates of the maximum and minimum values of the graph $y = \frac{x^3}{3} - \frac{x^2}{2} - 6x + \frac{5}{3}$ and distinguish between them. Sketch the graph.

Following the given procedure:

(i) Since $y = \frac{x^3}{3} - \frac{x^2}{2} - 6x + \frac{5}{3}$ then $\frac{dy}{dx} = x^2 - x - 6$

(ii) At a turning point, $\frac{dy}{dx} = 0$. Hence $x^2 - x - 6 = 0$, i.e. $(x+2)(x-3) = 0$, from which $x = -2$ or $x = 3$.

(iii) When $x = -2$,

$$y = \frac{(-2)^3}{3} - \frac{(-2)^2}{2} - 6(-2) + \frac{5}{3} = 9$$

When $x = 3$,

$$y = \frac{(3)^3}{3} - \frac{(3)^2}{2} - 6(3) + \frac{5}{3} = -11\frac{5}{6}$$

Thus the co-ordinates of the turning points are (−2, 9) and $\left(\mathbf{3, -11\tfrac{5}{6}}\right)$.

(iv) Since $\frac{dy}{dx} = x^2 - x - 6$ then $\frac{d^2y}{dx^2} = 2x - 1$.

When $x = -2$,

$$\frac{d^2y}{dx^2} = 2(-2) - 1 = -5,$$

which is negative.

Hence (−2, 9) is a maximum point.

When $x = 3$,

$$\frac{d^2y}{dx^2} = 2(3) - 1 = 5,$$

which is positive.

Hence $\left(3, -11\frac{5}{6}\right)$ is a minimum point.

Knowing $(-2, 9)$ is a maximum point (i.e. crest of a wave), and $\left(3, -11\frac{5}{6}\right)$ is a minimum point (i.e. bottom of a valley) and that when $x = 0$, $y = \frac{5}{3}$, a sketch may be drawn as shown in Fig. 11.

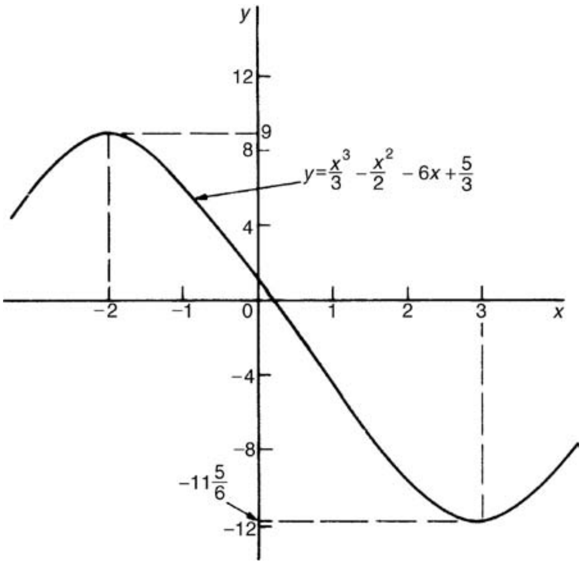

Figure 11

Problem 14. Determine the turning points on the curve $y = 4\sin x - 3\cos x$ in the range $x = 0$ to $x = 2\pi$ radians, and distinguish between them. Sketch the curve over one cycle.

Since $y = 4\sin x - 3\cos x$

then $\frac{dy}{dx} = 4\cos x + 3\sin x = 0$,

for a turning point, from which,

$$4\cos x = -3\sin x \text{ and}$$

$$\frac{-4}{3} = \frac{\sin x}{\cos x} = \tan x$$

Hence $x = \tan^{-1}\left(\frac{-4}{3}\right) = 126°52'$ or $306°52'$, since tangent is negative in the second and fourth quadrants.

When $x = 126°52'$,

$$y = 4\sin 126°52' - 3\cos 126°52' = 5$$

When $x = 306°52'$,

$$y = 4\sin 306°52' - 3\cos 306°52' = -5$$

$$126°52' = \left(125°52' \times \frac{\pi}{180}\right) \text{ radians}$$
$$= 2.214 \text{ rad}$$

$$306°52' = \left(306°52' \times \frac{\pi}{180}\right) \text{ radians}$$
$$= 5.356 \text{ rad}$$

Hence (2.214, 5) and (5.356, −5) are the co-ordinates of the turning points.

$$\frac{d^2y}{dx^2} = -4\sin x + 3\cos x$$

When $x = 2.214$ rad,

$$\frac{d^2y}{dx^2} = -4\sin 2.214 + 3\cos 2.214,$$

which is negative.

Hence (2.214, 5) is a maximum point.

When $x = 5.356$ rad,

$$\frac{d^2y}{dx^2} = -4\sin 5.356 + 3\cos 5.356,$$

which is positive.

Hence (5.356, −5) is a minimum point.

A sketch of $y = 4\sin x - 3\cos x$ is shown in Fig. 12.

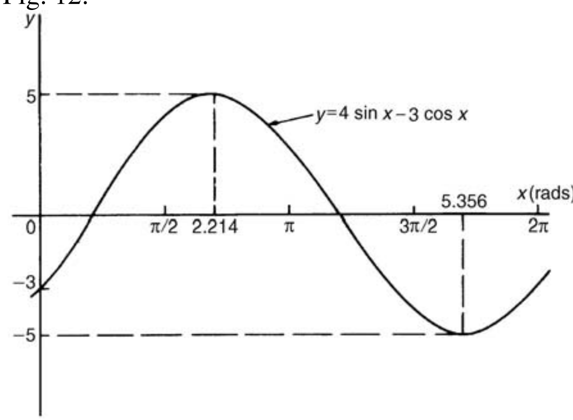

Figure 12

Exercise 8. Turning points

D. Maxima and Minima

There are many **practical problems** involving maximum and minimum values which occur in science and engineering. Usually, an equation has to be determined from given data, and rearranged where necessary, so that it contains only one variable. Some examples are demonstrated in Problems 15 to 20.

Problem 15. A rectangular area is formed having a perimeter of 40 cm. Determine the length and breadth of the rectangle if it is to enclose the maximum possible area.

Let the dimensions of the rectangle be x and y. Then the perimeter of the rectangle is $(2x + 2y)$. Hence

$$2x + 2y = 40,$$

$$\text{or} \quad x + y = 20 \qquad (1)$$

Since the rectangle is to enclose the maximum possible area, a formula for area A must be obtained in terms of one variable only.
Area $A = xy$. From equation (1), $x = 20 - y$
Hence, area $A = (20 - y)y = 20y - y^2$

$$\frac{\mathrm{d}A}{\mathrm{d}y} = 20 - 2y = 0$$

for a turning point, from which, $y = 10$ cm

$$\frac{\mathrm{d}^2A}{\mathrm{d}y^2} = -2,$$

which is negative, giving a maximum point.
When $y = 10$ cm, $x = 10$ cm, from equation (1).

Hence the length and breadth of the rectangle are each 10 cm, i.e. a square gives the maximum possible area. When the perimeter of a rectangle is 40 cm, the maximum possible area is $10 \times 10 =$ **100 cm^2**.

Problem 16. A rectangular sheet of metal having dimensions 20 cm by 12 cm has squares removed from each of the four corners and the sides bent upwards to form an open box. Determine the maximum possible volume of the box.

The squares to be removed from each corner are shown in Fig. 13, having sides x cm. When the sides are bent upwards the dimensions of the box will be:

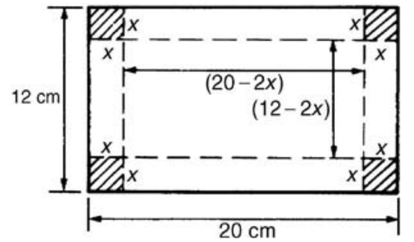

Figure 13

length $(20 - 2x)$ cm, breadth $(12 - 2x)$ cm and height, x cm.

Volume of box,

$$V = (20 - 2x)(12 - 2x)(x)$$

$$= 240x - 64x^2 + 4x^3$$

$$\frac{dV}{dx} = 240 - 128x + 12x^2 = 0$$

for a turning point

Hence $4(60 - 32x + 3x^2) = 0$,

i.e. $3x^2 - 32x + 60 = 0$

Using the quadratic formula,

$$x = \frac{32 \pm \sqrt{(-32)^2 - 4(3)(60)}}{2(3)}$$
$$= 8.239 \text{ cm or } 2.427 \text{ cm.}$$

Since the breadth is $(12 - 2x)$ cm then $x = 8.239$ cm is not possible and is neglected. Hence $x = 2.427$ cm

$$\frac{d^2V}{dx^2} = -128 + 24x.$$

When $x = 2.427$, $\frac{d^2V}{dx^2}$ is negative, giving a maximum value.

The dimensions of the box are:

length $= 20 - 2(2.427) = 15.146$ cm,

breadth $= 12 - 2(2.427) = 7.146$ cm,

and height $= 2.427$ cm

Maximum volume $= (15.146)(7.146)(2.427)$ $= \mathbf{262.7\ cm^3}$

Problem 17. Determine the height and radius of a cylinder of volume $200\,\text{cm}^3$ which has the least surface area.

Let the cylinder have radius r and perpendicular height h.

Volume of cylinder,

$$V = \pi r^2 h = 200 \qquad (1)$$

Surface area of cylinder,

$$A = 2\pi rh + 2\pi r^2$$

Least surface area means minimum surface area and a formula for the surface area in terms of one variable only is required.

From equation (1),

$$h = \frac{200}{\pi r^2} \qquad (2)$$

Hence surface area,

$$A = 2\pi r\left(\frac{200}{\pi r^2}\right) + 2\pi r^2$$
$$= \frac{400}{r} + 2\pi r^2 = 400r^{-1} + 2\pi r^2$$
$$\frac{dA}{dr} = \frac{-400}{r^2} + 4\pi r = 0,$$

for a turning point.

Hence $4\pi r = \frac{400}{r^2}$ and $r^3 = \frac{400}{4\pi}$,

from which,

$$r = \sqrt[3]{\left(\frac{100}{\pi}\right)} = 3.169 \text{ cm}$$
$$\frac{d^2A}{dr^2} = \frac{800}{r^3} + 4\pi.$$

When $r = 3.169$ cm, $\frac{d^2A}{dr^2}$ is positive, giving a minimum value.

From equation (2),

when $r = 3.169$ cm,

$$h = \frac{200}{\pi(3.169)^2} = 6.339 \text{ cm}$$

Hence for the least surface area, a cylinder of volume 200 cm^3 has a radius of 3.169 cm and height of 6.339 cm.

Problem 18. Determine the area of the largest piece of rectangular ground that can be enclosed by 100 m of fencing, if part of an existing straight wall is used as one side.

Let the dimensions of the rectangle be x and y as shown in Fig. 14, where PQ represents the straight wall.

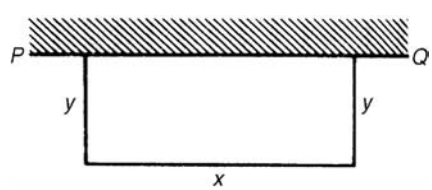

Figure 14

From Fig. 14,

$$x + 2y = 100 \qquad (1)$$

Area of rectangle,

$$A = xy \qquad (2)$$

Since the maximum area is required, a formula for area A is needed in terms of one variable only.
From equation (1), $x = 100 - 2y$
Hence area $A = xy = (100 - 2y)y = 100y - 2y^2$

$$\frac{dA}{dy} = 100 - 4y = 0,$$

for a turning point, from which, $y = 25$ m

$$\frac{d^2A}{dy^2} = -4,$$

which is negative, giving a maximum value.
When $y = 25$ m, $x = 50$ m from equation (1).
Hence the **maximum possible area** $= xy = (50)(25) = \mathbf{1250\,m^2}$.

Problem 19. An open rectangular box with square ends is fitted with an overlapping lid which covers the top and the front face. Determine the maximum volume of the box if 6 m^2 of metal are used in its construction.

A rectangular box having square ends of side x and length y is shown in Fig. 15.

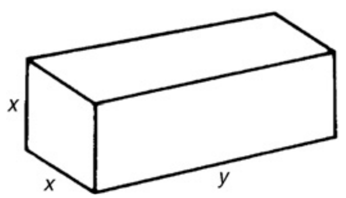

Figure 15

Surface area of box, A, consists of two ends and five faces (since the lid also covers the front face.) Hence

$$A = 2x^2 + 5xy = 6 \qquad (1)$$

Since it is the maximum volume required, a formula for the volume in terms of one variable only is needed. Volume of box, $V = x^2y$.

From equation (1),

$$y = \frac{6 - 2x^2}{5x} = \frac{6}{5x} - \frac{2x}{5} \qquad (2)$$

Hence volume

$$V = x^2y = x^2\left(\frac{6}{5x} - \frac{2x}{5}\right) = \frac{6x}{5} - \frac{2x^3}{5}$$

$$\frac{dV}{dx} = \frac{6}{5} - \frac{6x^2}{5} = 0$$

for a maximum or minimum value
Hence $6 = 6x^2$, giving $x = 1$ m ($x = -1$ is not possible, and is thus neglected).

$$\frac{d^2V}{dx^2} = \frac{-12x}{5}.$$

When $x = 1$, $\frac{d^2V}{dx^2}$ is negative, giving a maximum value.
From equation (2), when $x = 1$,

$$y = \frac{6}{5(1)} - \frac{2(1)}{5} = \frac{4}{5}$$

Hence the maximum volume of the box is given by

$$V = x^2y = (1)^2\left(\tfrac{4}{5}\right) = \mathbf{\tfrac{4}{5}\,m^3}$$

Problem 20. Find the diameter and height of a cylinder of maximum volume which can be cut from a sphere of radius 12 cm.

A cylinder of radius r and height h is shown enclosed in a sphere of radius $R = 12$ cm in Fig. 16. Volume of cylinder,

$$V = \pi r^2 h \qquad (1)$$

Using the right-angled triangle OPQ shown in Fig. 16,

$$r^2 + \left(\frac{h}{2}\right)^2 = R^2 \quad \text{by Pythagoras' theorem,}$$

i.e. $$r^2 + \frac{h^2}{4} = 144 \qquad (2)$$

Since the maximum volume is required, a formula for the volume V is needed in terms of one variable only. From equation (2),

$$r^2 = 144 - \frac{h^2}{4}$$

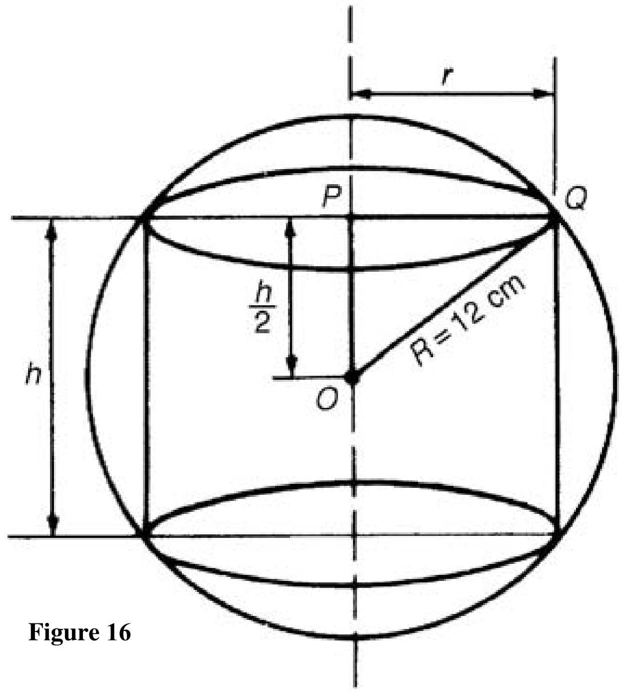

Figure 16

Substituting into equation (1) gives:

$$V = \pi\left(144 - \frac{h^2}{4}\right)h = 144\pi h - \frac{\pi h^3}{4}$$

$$\frac{dV}{dh} = 144\pi - \frac{3\pi h^2}{4} = 0,$$

for a maximum or minimum value.
Hence

$$144\pi = \frac{3\pi h^2}{4}$$

from which, $h = \sqrt{\frac{(144)(4)}{3}} = 13.86\,\text{cm}$

$$\frac{d^2V}{dh^2} = \frac{-6\pi h}{4}$$

When $h = 13.86$, $\frac{d^2V}{dh^2}$ is negative, giving a maximum value.
From equation (2),

$$r^2 = 144 - \frac{h^2}{4} = 144 - \frac{13.86^2}{4}$$

from which, radius $r = 9.80\,\text{cm}$

Diameter of cylinder $= 2r = 2(9.80) = 19.60\,\text{cm}$.

Hence the cylinder having the maximum volume that can be cut from a sphere of radius 12 cm is one in which the diameter is 19.60 cm and the height is 13.86 cm.

Exercise 9. Maxima and minima

E. Tangents and Normals

Tangents

The equation of the tangent to a curve $y = f(x)$ at the point (x_1, y_1) is given by:

$$y - y_1 = m(x - x_1)$$

where $m = \frac{dy}{dx}$ = gradient of the curve at (x_1, y_1).

Problem 21. Find the equation of the tangent to the curve $y = x^2 - x - 2$ at the point (1, −2).

Gradient, m

$$= \frac{dy}{dx} = 2x - 1$$

At the point (1, −2), $x = 1$ and $m = 2(1) - 1 = 1$.
Hence the equation of the tangent is:

$$y - y_1 = m(x - x_1)$$
i.e. $$y - (-2) = 1(x - 1)$$
i.e. $$y + 2 = x - 1$$
or $$\mathbf{y = x - 3}$$

The graph of $y = x^2 - x - 2$ is shown in Fig. 17 The line AB is the tangent to the curve at the point C, i.e. (1, −2), and the equation of this line is $y = x - 3$.

Normals

The normal at any point on a curve is the line which passes through the point and is at right angles to the tangent. Hence, in Fig. 17, the line CD is the normal.

It may be shown that if two lines are at right angles then the product of their gradients is −1. Thus if m is the gradient of the tangent, then the gradient of the normal is $-\frac{1}{m}$

Hence the equation of the normal at the point (x_1, y_1) is given by:

$$y - y_1 = -\frac{1}{m}(x - x_1)$$

Problem 22. Find the equation of the normal to the curve $y = x^2 - x - 2$ at the point (1, −2).

$m = 1$ from Problem 21, hence the equation of the normal is

$$y - y_1 = -\frac{1}{m}(x - x_1)$$
i.e. $$y - (-2) = -\frac{1}{1}(x - 1)$$
i.e. $$y + 2 = -x + 1$$
or $$\mathbf{y = -x - 1}$$

Thus the line CD in Fig. 17 has the equation $y = -x - 1$.

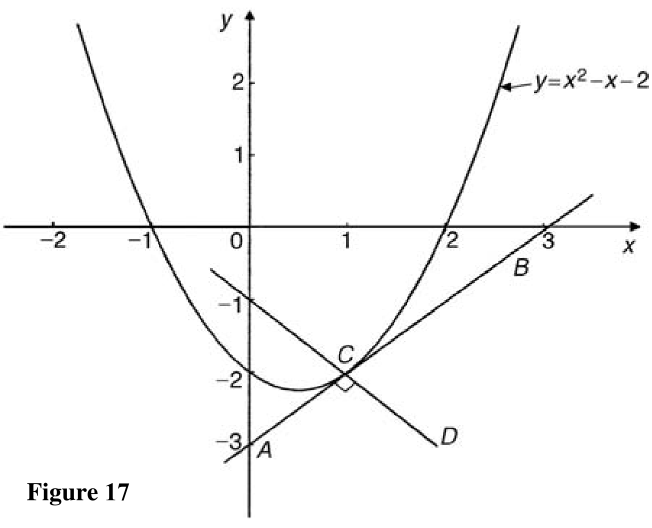

Figure 17

Problem 23. Determine the equations of the tangent and normal to the curve $y = \dfrac{x^3}{5}$ at the point $\left(-1, -\dfrac{1}{5}\right)$

Gradient m of curve $y = \dfrac{x^3}{5}$ is given by

$$m = \frac{dy}{dx} = \frac{3x^2}{5}$$

At the point $\left(-1, -\frac{1}{5}\right)$, $x = -1$ and $m = \dfrac{3(-1)^2}{5} = \dfrac{3}{5}$

Equation of the tangent is:

$$y - y_1 = m(x - x_1)$$

i.e. $y - \left(-\dfrac{1}{5}\right) = \dfrac{3}{5}(x - (-1))$

i.e. $y + \dfrac{1}{5} = \dfrac{3}{5}(x + 1)$

or $5y + 1 = 3x + 3$

or $\mathbf{5y - 3x = 2}$

Equation of the normal is:

$$y - y_1 = -\frac{1}{m}(x - x_1)$$

i.e. $y - \left(-\dfrac{1}{5}\right) = \dfrac{-1}{(3/5)}(x - (-1))$

i.e. $y + \dfrac{1}{5} = -\dfrac{5}{3}(x + 1)$

i.e. $y + \dfrac{1}{5} = -\dfrac{5}{3}x - \dfrac{5}{3}$

Multiplying each term by 15 gives:

$$15y + 3 = -25x - 25$$

Hence **equation of the normal** is:

$$\mathbf{15y + 25x + 28 = 0}$$

Exercise 10. Tangents and Normals

F. Small Changes

If y is a function of x, i.e. $y=f(x)$, and the approximate change in y corresponding to a small change δx in x is required, then:

$$\frac{\delta y}{\delta x} \approx \frac{dy}{dx}$$

and $\boldsymbol{\delta y \approx \frac{dy}{dx} \cdot \delta x}$ or $\boldsymbol{\delta y \approx f'(x) \cdot \delta x}$

Problem 24. Given $y=4x^2-x$, determine the approximate change in y if x changes from 1 to 1.02.

Since $y=4x^2-x$, then

$$\frac{dy}{dx} = 8x-1$$

Approximate change in y,

$$\delta y \approx \frac{dy}{dx} \cdot \delta x \approx (8x-1)\delta x$$

When $x=1$ and $\delta x = 0.02$, $\boldsymbol{\delta y} \approx [8(1)-1](0.02)$
$\approx \mathbf{0.14}$

[Obviously, in this case, the exact value of dy may be obtained by evaluating y when $x=1.02$, i.e. $y=4(1.02)^2-1.02=3.1416$ and then subtracting from it the value of y when $x=1$, i.e. $y=4(1)^2-1=3$, giving $\boldsymbol{\delta y}=3.1416-3=\mathbf{0.1416}$.

Using $\delta y = \frac{dy}{dx} \cdot \delta x$ above gave 0.14, which shows that the formula gives the approximate change in y for a small change in x.]

Problem 25. The time of swing T of a pendulum is given by $T=k\sqrt{l}$, where k is a constant. Determine the percentage change in the time of swing if the length of the pendulum l changes from 32.1 cm to 32.0 cm.

If $T=k\sqrt{l}=kl^{\frac{1}{2}}$, then

$$\frac{dT}{dl} = k\left(\frac{1}{2}l^{\frac{-1}{2}}\right) = \frac{k}{2\sqrt{l}}$$

Approximate change in T,

$$\delta t \approx \frac{dT}{dl}\delta l \approx \left(\frac{k}{2\sqrt{l}}\right)\delta l$$
$$\approx \left(\frac{k}{2\sqrt{l}}\right)(-0.1)$$

(negative since l decreases)
Percentage error

$$= \left(\frac{\text{approximate change in } T}{\text{original value of } T}\right)100\%$$
$$= \frac{\left(\frac{k}{2\sqrt{l}}\right)(-0.1)}{k\sqrt{l}} \times 100\%$$
$$= \left(\frac{-0.1}{2l}\right)100\% = \left(\frac{-0.1}{2(32.1)}\right)100\%$$
$$= \mathbf{-0.156\%}$$

Hence the change in the time of swing is a decrease of 0.156%.

Problem 26. A circular template has a radius of 10 cm (± 0.02). Determine the possible error in calculating the area of the template. Find also the percentage error.

Area of circular template, $A=\pi r^2$, hence

$$\frac{dA}{dr} = 2\pi r$$

Approximate change in area,

$$\delta A \approx \frac{dA}{dr} \cdot \delta r \approx (2\pi r)\delta r$$

When $r=10$ cm and $\delta r=0.02$,

$\delta A=(2\pi 10)(0.02)\approx 0.4\pi$ cm^2

i.e. **the possible error in calculating the template area is approximately 1.257 cm^2.**

$$\textbf{Percentage error} \approx \left(\frac{0.4\pi}{\pi(10)^2}\right)100\%$$
$$= \mathbf{0.40\%}$$

Exercise 11. Small changes