Module 2
Methods of Differentiation

A. Differentiation of common functions

The **standard derivatives** are true for all real values of x.

y or $f(x)$	$\frac{dy}{dx}$ of $f'(x)$
ax^n	ax^{n-1}
$\sin ax$	$a\cos ax$
$\cos ax$	$-a\sin ax$
e^{ax}	ae^{ax}
$\ln ax$	$\frac{1}{x}$

The **differential coefficient of a sum or difference** is the sum or difference of the differential coefficients of the separate terms.
Thus, if $f(x) = p(x) + q(x) - r(x)$, (where f, p, q and r are functions), then $f'(x) = p'(x) + q'(x) - r'(x)$

Problem 1. Find the differential coefficients of:
(a) $y = 12x^3$ (b) $y = \frac{12}{x^3}$

If $y = ax^n$ then $\frac{dy}{dx} = anx^{n-1}$

(a) Since $y = 12x^3$, $a = 12$ and $n = 3$ thus $\frac{dy}{dx} = (12)\ (3)x^{3-1} = \mathbf{36x^2}$

(b) $y = \frac{12}{x^3}$ is rewritten in the standard ax^n form as $y = 12x^{-3}$ and in the general rule $a = 12$ and $n = -3$

Thus $\frac{dy}{dx} = (12)(-3)x^{-3-1}$

$= -36x^{-4} = -\frac{\mathbf{36}}{\mathbf{x^4}}$

Problem 2. Differentiate: (a) $y = 6$ (b) $y = 6x$

(a) $y = 6$ may be written as $y = 6x^0$, i.e. in the general rule $a = 6$ and $n = 0$.

Hence $\frac{dy}{dx} = (6)(0)x^{0-1} = \mathbf{0}$

In general, **the differential coefficient of a constant is always zero**.

(b) Since $y = 6x$, in the general rule $a = 6$ and $n = 1$

Hence $\frac{dy}{dx} = (6)(1)x^{1-1} = 6x^0 = \mathbf{6}$

In general, the differential coefficient of kx, where k is a constant, is always k.

Problem 3. Find the derivatives of:
(a) $y = 3\sqrt{x}$ (b) $y = \frac{5}{\sqrt[3]{x^4}}$

(a) $y = 3\sqrt{x}$ is rewritten in the standard differential form as $y = 3x^{1/2}$

In the general rule, $a = 3$ and $n = \frac{1}{2}$

Thus $\frac{dy}{dx} = (3)\left(\frac{1}{2}\right)x^{\frac{1}{2}-1} = \frac{3}{2}x^{-\frac{1}{2}}$

$= \frac{3}{2x^{1/2}} = \frac{\mathbf{3}}{\mathbf{2\sqrt{x}}}$

(b) $y = \frac{5}{\sqrt[3]{x^4}} = \frac{5}{x^{4/3}} = 5x^{-4/3}$

In the general rule, $a = 5$ and $n = -\frac{4}{3}$

Thus $\frac{dy}{dx} = (5)\left(-\frac{4}{3}\right)x^{(-4/3)-1}$

$$= \frac{-20}{3}x^{-7/3} = \frac{-20}{3x^{7/3}} = \mathbf{\frac{-20}{3\sqrt[3]{x^7}}}$$

Problem 4. Differentiate:
$y = 5x^4 + 4x - \frac{1}{2x^2} + \frac{1}{\sqrt{x}} - 3$ with respect to x

$y = 5x^4 + 4x - \frac{1}{2x^2} + \frac{1}{\sqrt{x}} - 3$ is rewritten as

$y = 5x^4 + 4x - \frac{1}{2}x^{-2} + x^{-1/2} - 3$

When differentiating a sum, each term is differentiated in turn.

Thus $\frac{dy}{dx} = (5)(4)x^{4-1} + (4)(1)x^{1-1} - \frac{1}{2}(-2)x^{-2-1} + (1)\left(-\frac{1}{2}\right)x^{(-1/2)-1} - 0$

$$= 20x^3 + 4 + x^{-3} - \frac{1}{2}x^{-3/2}$$

i.e. $\mathbf{\frac{dy}{dx} = 20x^3 + 4 - \frac{1}{x^3} - \frac{1}{2\sqrt{x^3}}}$

Problem 5. Find the differential coefficients of: (a) $y = 3\sin 4x$ (b) $f(t) = 2\cos 3t$ with respect to the variable

(a) When $y = 3\sin 4x$ then $\frac{dy}{dx} = (3)(4\cos 4x)$
$= \mathbf{12\cos 4x}$

(b) When $f(t) = 2\cos 3t$ then
$f'(t) = (2)(-3\sin 3t) = \mathbf{-6\sin 3t}$

Problem 6. Determine the derivatives of:
(a) $y = 3e^{5x}$ (b) $f(\theta) = \frac{2}{e^{3\theta}}$ (c) $y = 6\ln 2x$

(a) When $y = 3e^{5x}$ then $\frac{dy}{dx} = (3)(5)e^{5x} = \mathbf{15e^{5x}}$

(b) $f(\theta) = \frac{2}{e^{3\theta}} = 2e^{-3\theta}$, thus

$f'(\theta) = (2)(-3)e^{-3\theta} = -6e^{-3\theta} = \mathbf{\frac{-6}{e^{3\theta}}}$

(c) When $y = 6\ln 2x$ then $\frac{dy}{dx} = 6\left(\frac{1}{x}\right) = \mathbf{\frac{6}{x}}$

Problem 7. Find the gradient of the curve $y = 3x^4 - 2x^2 + 5x - 2$ at the points $(0, -2)$ and $(1, 4)$

The gradient of a curve at a given point is given by the corresponding value of the derivative. Thus, since $y = 3x^4 - 2x^2 + 5x - 2$ then the gradient $= \frac{dy}{dx} = 12x^3 - 4x + 5$

At the point $(0, -2)$, $x = 0$.

Thus the gradient $= 12(0)^3 - 4(0) + 5 = \mathbf{5}$

At the point $(1, 4)$, $x = 1$.

Thus the gradient $= 12(1)^3 - 4(1) + 5 = \mathbf{13}$

Problem 8. Determine the co-ordinates of the point on the graph $y = 3x^2 - 7x + 2$ where the gradient is -1

The gradient of the curve is given by the derivative.

When $y = 3x^2 - 7x + 2$ then $\frac{dy}{dx} = 6x - 7$

Since the gradient is -1 then $6x - 7 = -1$, from which, $x = 1$

When $x = 1$, $y = 3(1)^2 - 7(1) + 2 = -2$

Hence the gradient is -1 at the point $(1, -2)$

Exercise 7. Differentiating common functions

B. Differentiation of a product

When $y = uv$, and u and v are both functions of x,

then $$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

This is known as the **product rule**.

Problem 9. Find the differential coefficient of:
$$y = 3x^2 \sin 2x$$

$3x^2 \sin 2x$ is a product of two terms $3x^2$ and $\sin 2x$. Let $u = 3x^2$ and $v = \sin 2x$

Using the product rule:

$$\frac{dy}{dx} = u \quad \frac{dv}{dx} \quad + \quad v \quad \frac{du}{dx}$$
$$\downarrow \quad \downarrow \quad \downarrow \quad \downarrow$$

gives: $\dfrac{dy}{dx} = (3x^2)(2\cos 2x) + (\sin 2x)(6x)$

i.e. $\dfrac{dy}{dx} = 6x^2 \cos 2x + 6x \sin 2x$

$= \mathbf{6x(x\cos 2x + \sin 2x)}$.

Note that the differential coefficient of a product is **not** obtained by merely differentiating each term and multiplying the two answers together. The product rule formula **must** be used when differentiating products.

Problem 10. Find the rate of change of y with respect to x given: $y = 3\sqrt{x}\ln 2x$

The rate of change of y with respect to x is given by $\dfrac{dy}{dx}$.

$y = 3\sqrt{x}\ln 2x = 3x^{1/2}\ln 2x$, which is a product.

Let $u = 3x^{1/2}$ and $v = \ln 2x$

Then $$\frac{dy}{dx} = u \quad \frac{dv}{dx} \quad + \quad v \quad \frac{du}{dx}$$
$$\downarrow \quad \downarrow \quad \downarrow \quad \downarrow$$

$$= (3x^{1/2})\left(\frac{1}{x}\right) + (\ln 2x)\left[3\left(\frac{1}{2}\right)x^{(1/2)-1}\right]$$

$$= 3x^{(1/2)-1} + (\ln 2x)\left(\frac{3}{2}\right)x^{-1/2}$$

$$= 3x^{-1/2}\left(1 + \frac{1}{2}\ln 2x\right)$$

i.e. $$\boldsymbol{\frac{dy}{dx} = \frac{3}{\sqrt{x}}\left(1 + \frac{1}{2}\ln 2x\right)}$$

Problem 11. Differentiate: $y = x^3 \cos 3x \ln x$

Let $u = x^3 \cos 3x$ (i.e. a product) and $v = \ln x$

Then $\dfrac{dy}{dx} = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$

where $\dfrac{du}{dx} = (x^3)(-3\sin 3x) + (\cos 3x)(3x^2)$

and $\dfrac{dv}{dx} = \dfrac{1}{x}$

Hence $\dfrac{dy}{dx} = (x^3\cos 3x)\left(\dfrac{1}{x}\right)$

$+ (\ln x)[-3x^3 \sin 3x + 3x^2 \cos 3x]$

$= x^2 \cos 3x + 3x^2 \ln x(\cos 3x - x\sin 3x)$

i.e. $\boldsymbol{\dfrac{dy}{dx} = x^2\{\cos 3x + 3\ln x(\cos 3x - x\sin 3x)\}}$

Problem 12. Determine the rate of change of voltage, given $v = 5t \sin 2t$ volts, when $t = 0.2$ s

Rate of change of voltage

$$= \frac{dv}{dt} = (5t)(2\cos 2t) + (\sin 2t)(5)$$
$$= 10t\cos 2t + 5\sin 2t$$

When $t = 0.2$,

$$\frac{dv}{dt} = 10(0.2)\cos 2(0.2) + 5\sin 2(0.2)$$
$$= 2\cos 0.4 + 5\sin 0.4$$

(where cos 0.4 means the cosine of 0.4 radians = 0.92106)

Hence $\dfrac{dv}{dt} = 2(0.92106) + 5(0.38942)$

$= 1.8421 + 1.9471 = 3.7892$

i.e. **the rate of change of voltage when $t = 0.2$ s is 3.79 volts/s, correct to 3 significant figures.**

Exercise 8. Differentiating products

C. Differentiation of a quotient

When $y = \frac{u}{v}$, and u and v are both functions of x

then $$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

This is known as the **quotient rule**.

Problem 13. Find the differential coefficient of: $y = \frac{4\sin 5x}{5x^4}$

$\frac{4\sin 5x}{5x^4}$ is a quotient. Let $u = 4\sin 5x$ and $v = 5x^4$

(Note that v is **always** the denominator and u the numerator)

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

where $\frac{du}{dx} = (4)(5)\cos 5x = 20\cos 5x$

and $\frac{dv}{dx} = (5)(4)x^3 = 20x^3$

Hence $$\frac{dy}{dx} = \frac{(5x^4)(20\cos 5x) - (4\sin 5x)(20x^3)}{(5x^4)^2}$$

$$= \frac{100x^4\cos 5x - 80x^3\sin 5x}{25x^8}$$

$$= \frac{20x^3[5x\cos 5x - 4\sin 5x]}{25x^8}$$

i.e. $$\boldsymbol{\frac{dy}{dx} = \frac{4}{5x^5}(5x\cos 5x - 4\sin 5x)}$$

Note that the differential coefficient is **not** obtained by merely differentiating each term in turn and then dividing the numerator by the denominator. The quotient formula **must** be used when differentiating quotients.

Problem 14. Determine the differential coefficient of: $y = \tan ax$

$y = \tan ax = \frac{\sin ax}{\cos ax}$. Differentiation of $\tan ax$ is thus treated as a quotient with $u = \sin ax$ and $v = \cos ax$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$= \frac{(\cos ax)(a\cos ax) - (\sin ax)(-a\sin ax)}{(\cos ax)^2}$$

$$= \frac{a\cos^2 ax + a\sin^2 ax}{(\cos ax)^2}$$

$$= \frac{a(\cos^2 ax + \sin^2 ax)}{\cos^2 ax}$$

$$= \frac{a}{\cos^2 ax} \text{ since } \cos^2 ax + \sin^2 ax = 1$$

Hence $\boldsymbol{\frac{dy}{dx} = a\sec^2 ax}$ since $\sec^2 ax = \frac{1}{\cos^2 ax}$

Problem 15. Find the derivative of: $y = \sec ax$

$y = \sec ax = \frac{1}{\cos ax}$ (i.e. a quotient), Let $u = 1$ and $v = \cos ax$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$= \frac{(\cos ax)(0) - (1)(-a\sin ax)}{(\cos ax)^2}$$

$$= \frac{a\sin ax}{\cos^2 ax} = a\left(\frac{1}{\cos ax}\right)\left(\frac{\sin ax}{\cos ax}\right)$$

i.e. $$\boldsymbol{\frac{dy}{dx} = a\sec ax\tan ax}$$

Problem 16. Differentiate: $y = \frac{te^{2t}}{2\cos t}$

The function $\dfrac{te^{2t}}{2\cos t}$ is a quotient, whose numerator is a product.

Let $u = te^{2t}$ and $v = 2\cos t$ then

$\dfrac{du}{dt} = (t)(2e^{2t}) + (e^{2t})(1)$ and $\dfrac{dv}{dt} = -2\sin t$

$$\text{Hence}\quad \frac{dy}{dx} = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$$

$$= \frac{(2\cos t)[2te^{2t} + e^{2t}] - (te^{2t})(-2\sin t)}{(2\cos t)^2}$$

$$= \frac{4te^{2t}\cos t + 2e^{2t}\cos t + 2te^{2t}\sin t}{4\cos^2 t}$$

$$= \frac{2e^{2t}[2t\cos t + \cos t + t\sin t]}{4\cos^2 t}$$

$$\text{i.e.}\quad \mathbf{\frac{dy}{dx} = \frac{e^{2t}}{2\cos^2 t}(2t\cos t + \cos t + t\sin t)}$$

Problem 17. Determine the gradient of the curve $y = \dfrac{5x}{2x^2 + 4}$ at the point $\left(\sqrt{3}, \dfrac{\sqrt{3}}{2}\right)$

Let $y = 5x$ and $v = 2x^2 + 4$

$$\frac{dy}{dx} = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2} = \frac{(2x^2+4)(5) - (5x)(4x)}{(2x^2+4)^2}$$

$$= \frac{10x^2 + 20 - 20x^2}{(2x^2+4)^2} = \frac{20 - 10x^2}{(2x^2+4)^2}$$

At the point $\left(\sqrt{3}, \dfrac{\sqrt{3}}{2}\right)$, $x = \sqrt{3}$,

$$\text{hence the gradient} = \frac{dy}{dx} = \frac{20 - 10(\sqrt{3})^2}{[2(\sqrt{3})^2 + 4]^2}$$

$$= \frac{20 - 30}{100} = \mathbf{-\frac{1}{10}}$$

Exercise 9. Differentiating quotients

D. Function of a function

It is often easier to make a substitution before differentiating.

If y is a function of x then $\mathbf{\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}}$

This is known as the **'function of a function'** rule (or sometimes the **chain rule**).

For example, if $y = (3x-1)^9$ then, by making the substitution $u = (3x-1)$, $y = u^9$, which is of the 'standard' from.

Hence $\dfrac{dy}{du} = 9u^8$ and $\dfrac{du}{dx} = 3$

Then $\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx} = (9u^8)(3) = 27u^8$

Rewriting u as $(3x-1)$ gives: $\mathbf{\dfrac{dy}{dx} = 27(3x-1)^8}$

Since y is a function of u, and u is a function of x, then y is a function of a function of x.

Problem 18. Differentiate: $y = 3\cos(5x^2 + 2)$

Let $u = 5x^2 + 2$ then $y = 3\cos u$

Hence $\frac{du}{dx} = 10x$ and $\frac{dy}{du} = -3\sin u$

Using the function of a function rule,

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = (-3\sin u)(10x) = -30x\sin u$$

Rewriting u as $5x^2 + 2$ gives:

$$\boldsymbol{\frac{dy}{dx} = -30x\sin(5x^2 + 2)}$$

Problem 19. Find the derivative of:
$$y = (4t^3 - 3t)^6$$

Let $u = 4t^3 - 3t$, then $y = u^6$

Hence $\frac{du}{dt} = 12t^2 - 3$ and $\frac{dy}{dt} = 6u^5$

Using the function of a function rule,

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = (6u^5)(12t^2 - 3)$$

Rewriting u as $(4t^3 - 3t)$ gives:

$$\frac{dy}{dt} = 6(4t^3 - 3t)^5(12t^2 - 3)$$
$$= \mathbf{18(4t^2 - 1)(4t^3 - 3t)^5}$$

Problem 20. Determine the differential coefficient of: $y = \sqrt{3x^2 + 4x - 1}$

$y = \sqrt{3x^2 + 4x - 1} = (3x^2 + 4x - 1)^{1/2}$

Let $u = 3x^2 + 4x - 1$ then $y = u^{1/2}$

Hence $\frac{du}{dx} = 6x + 4$ and $\frac{dy}{du} = \frac{1}{2}u^{-1/2} = \frac{1}{2\sqrt{u}}$

Using the function of a function rule,

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \left(\frac{1}{2\sqrt{u}}\right)(6x + 4) = \frac{3x + 2}{\sqrt{u}}$$

i.e. $\boldsymbol{\frac{dy}{dx} = \frac{3x + 2}{\sqrt{3x^2 + 4x - 1}}}$

Problem 21. Differentiate: $y = 3\tan^4 3x$

Let $u = \tan 3x$ then $y = 3u^4$

Hence $\frac{du}{dx} = 3\sec^2 3x$

and $\frac{dy}{du} = 12u^3$

Then $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = (12u^3)(3\sec^2 3x)$

$= 12(\tan 3x)^3(3\sec^2 3x)$

i.e. $\boldsymbol{\frac{dy}{dx} = 36\tan^3 3x\sec^2 3x}$

Problem 22. Find the differential coefficient of:
$$y = \frac{2}{(2t^3 - 5)^4}$$

$y = \frac{2}{(2t^3 - 5)^4} = 2(2t^3 - 5)^{-4}$. Let $u = (2t^3 - 5)$, then $y = 2u^{-4}$

Hence $\frac{du}{dt} = 6t^2$ and $\frac{dy}{du} = -8u^{-5} = \frac{-8}{u^5}$

Then $\frac{dy}{dt} = \frac{dy}{du} \times \frac{du}{dt} = \left(\frac{-8}{u^5}\right)(6t^2) = \boldsymbol{\frac{-48t^2}{(2t^3 - 5)^5}}$

Exercise 10. Function of a function

E. Successive differentiation

When a function $y = f(x)$ is differentiated with respect to x the differential coefficient is written as $\frac{dy}{dx}$ or $f'(x)$. If the expression is differentiated again, the second differential coefficient is obtained and is written as $\frac{d^2y}{dx^2}$ (pronounced dee two y by dee x squared) or $f''(x)$ (pronounced f double–dash x). By successive differentiation further higher derivatives such as $\frac{d^3y}{dx^3}$ and $\frac{d^4y}{dx^4}$ may be obtained.

Thus if $y = 3x^4$,

$$\frac{dy}{dx} = 12x^3, \quad \frac{d^2y}{dx^2} = 36x^2,$$

$$\frac{d^3y}{dx^3} = 72x, \quad \frac{d^4y}{dx^4} = 72 \text{ and } \frac{d^5y}{dx^5} = 0$$

Problem 23. If $f(x) = 2x^5 - 4x^3 + 3x - 5$, find $f''(x)$

$$f(x) = 2x^5 - 4x^3 + 3x - 5$$

$$f'(x) = 10x^4 - 12x^2 + 3$$

$$\boldsymbol{f''(x)} = 40x^3 - 24x = \mathbf{4x(10x^2 - 6)}$$

Problem 24. If $y = \cos x - \sin x$, evaluate x, in the range $0 \le x \le \frac{\pi}{2}$, when $\frac{d^2y}{dx^2}$ is zero

Since $y = \cos x - \sin x$, $\frac{dy}{dx} = -\sin x - \cos x$ and $\frac{d^2y}{dx^2} = -\cos x + \sin x$

When $\frac{d^2y}{dx^2}$ is zero, $-\cos x + \sin x = 0$,

i.e. $\sin x = \cos x$ or $\frac{\sin x}{\cos x} = 1$

Hence $\tan x = 1$ and $\boldsymbol{x} = \tan^{-1} 1 = \mathbf{45°}$ **or** $\boldsymbol{\frac{\pi}{4}}$ **rads** in the range $0 \le x \le \frac{\pi}{2}$

Problem 25. Given $y = 2xe^{-3x}$ show that

$$\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 9y = 0$$

$y = 2xe^{-3x}$ (i.e. a product)

Hence
$$\frac{dy}{dx} = (2x)(-3e^{-3x}) + (e^{-3x})(2)$$
$$= -6xe^{-3x} + 2e^{-3x}$$
$$\frac{d^2y}{dx^2} = [(-6x)(-3e^{-3x}) + (e^{-3x})(-6)] + (-6e^{-3x})$$
$$= 18xe^{-3x} - 6e^{-3x} - 6e^{-3x}$$

i.e.
$$\frac{d^2y}{dx^2} = 18xe^{-3x} - 12e^{-3x}$$

Substituting values into $\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 9y$ gives:

$$(18xe^{-3x} - 12e^{-3x}) + 6(-6xe^{-3x} + 2e^{-3x}) + 9(2xe^{-3x})$$
$$= 18xe^{-3x} - 12e^{-3x} - 36xe^{-3x} + 12e^{-3x} + 18xe^{-3x} = 0$$

Thus when $y = 2xe^{-3x}$, $\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 9y = 0$

Problem 26. Evaluate $\frac{d^2y}{d\theta^2}$ when $\theta = 0$ given: $y = 4 \sec 2\theta$

Since $y = 4 \sec 2\theta$, then

$$\frac{dy}{d\theta} = (4)(2)\sec 2\theta \tan 2\theta \text{ (from Problem 15)}$$
$$= 8 \sec 2\theta \tan 2\theta \text{ (i.e. a product)}$$
$$\frac{d^2y}{d\theta^2} = (8\sec 2\theta)(2\sec^2 2\theta) + (\tan 2\theta)[(8)(2)\sec 2\theta \tan 2\theta]$$
$$= 16\sec^3 2\theta + 16\sec 2\theta \tan^2 2\theta$$

When $\theta = 0$,

$$\frac{d^2y}{d\theta^2} = 16\sec^3 0 + 16\sec 0 \tan^2 0$$
$$= 16(1) + 16(1)(0) = \mathbf{16}$$

Exercise 11. Successive differentiation