Module 15
Second Moments of Area

A. Second moments of area and radius of gyration

The **first moment of area** about a fixed axis of a lamina of area A, perpendicular distance y from the centroid of the lamina is defined as Ay cubic units. The **second moment of area** of the same lamina as above is given by Ay^2, i.e. the perpendicular distance from the centroid of the area to the fixed axis is squared. Second moments of areas are usually denoted by I and have limits of mm^4, cm^4, and so on.

B. Radius of gyration

Several areas, $a_1, a_2, a_3, \ldots$ at distances $y_1, y_2, y_3, \ldots$ from a fixed axis, may be replaced by a single area A, where $A = a_1 + a_2 + a_3 + \cdots$ at distance k from the axis, such that $Ak^2 = \sum ay^2$. k is called the **radius of gyration** of area A about the given axis. Since $Ak^2 = \sum ay^2 = I$ then the radius of gyration, $\boldsymbol{k = \sqrt{\frac{I}{A}}}$

The second moment of area is a quantity much used in the theory of bending of beams, in the torsion of shafts, and in calculations involving water planes and centres of pressure.

C. Second moment of area of regular sections

The procedure to determine the second moment of area of regular sections about a given axis is (i) to find the second moment of area of a typical element and (ii) to sum all such second moments of area by integrating between appropriate limits.

For example, the second moment of area of the rectangle shown in Fig. 35 about axis PP is found by initially considering an elemental strip of width δx, par-allel to and distance x from axis PP. Area of shaded strip $= b\delta x$. Second moment of area of the shaded strip about $PP = (x^2)(b\delta x)$.

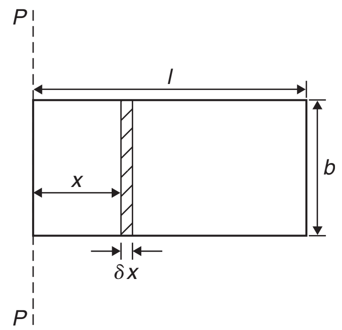

Figure 35

The second moment of area of the whole rectangle about PP is obtained by summing all such strips between $x = 0$ and $x = l$, i.e. $\sum_{x=0}^{x=l} x^2 b\delta x$. It is a fundamental theorem of integration that

$$\underset{\delta x \to 0}{\text{limit}} \sum_{x=0}^{x=l} x^2 b\delta x = \int_0^l x^2 b\, dx$$

Thus the second moment of area of the rectangle about

$$PP = b\int_0^l x^2 dx = b\left[\frac{x^3}{3}\right]_0^l = \boldsymbol{\frac{bl^3}{3}}$$

Since the total area of the rectangle, $A = lb$, then

$$I_{pp} = (lb)\left(\frac{l^2}{3}\right) = \boldsymbol{\frac{Al^2}{3}}$$

$$I_{pp} = Ak_{pp}^2 \text{ thus } k_{pp}^2 = \frac{l^2}{3}$$

i.e. the radius of gyration about axes PP,

$$\boldsymbol{k_{pp}} = \sqrt{\frac{l^2}{3}} = \boldsymbol{\frac{l}{\sqrt{3}}}$$

D. Parallel axis theorem

In Fig. 36, axis GG passes through the centroid C of area A. Axes DD and GG are in the same plane, are parallel to each other and distance d apart. The parallel axis theorem states:

$$\boldsymbol{I_{DD} = I_{GG} + Ad^2}$$

Using the parallel axis theorem the second moment of area of a rectangle about an axis through the centroid may be determined. In the rectangle shown in Fig. 37, $I_{pp} = \frac{bl^3}{3}$ (from above). From the parallel axis theorem $I_{pp} = I_{GG} + (bl)\left(\frac{l}{2}\right)^2$

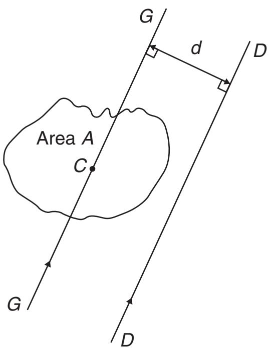

Figure 36

i.e. $\quad \frac{bl^3}{3} = I_{GG} + \frac{bl^3}{4}$

from which, $\quad \boldsymbol{I_{GG}} = \frac{bl^3}{3} - \frac{bl^3}{4} = \boldsymbol{\frac{bl^3}{12}}$

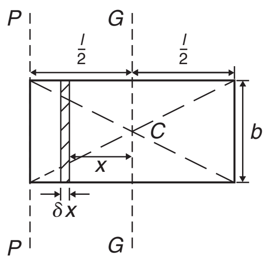

Figure 37

Perpendicular axis theorem

In Fig. 38, axes OX, OY and OZ are mutually perpendicular. If OX and OY lie in the plane of area A then the

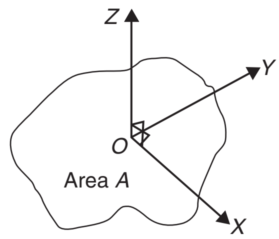

Figure 38

perpendicular axis theorem states:

$$\boldsymbol{I_{OZ} = I_{OX} + I_{OY}}$$

Summary of derived results

A summary of derive standard results for the second moment of area and radius of gyration of regular sections are listed in Table 3.

Table 3 Summary of standard results of the second moments of areas of regular sections

Shape	Position of axis	Second moment of area, I	Radius of gyration, k
Rectangle length l breadth b	(1) Coinciding with b	$\frac{bl^3}{3}$	$\frac{l}{\sqrt{3}}$
	(2) Coinciding with l	$\frac{lb^3}{3}$	$\frac{b}{\sqrt{3}}$
	(3) Through centroid, parallel to b	$\frac{bl^3}{12}$	$\frac{l}{\sqrt{12}}$
	(4) Through centroid, parallel to l	$\frac{lb^3}{12}$	$\frac{b}{\sqrt{12}}$
Triangle Perpendicular height h base b	(1) Coinciding with b	$\frac{bh^3}{12}$	$\frac{h}{\sqrt{6}}$
	(2) Through centroid, parallel to base	$\frac{bh^3}{36}$	$\frac{h}{\sqrt{18}}$
	(3) Through vertex, parallel to base	$\frac{bh^3}{4}$	$\frac{h}{\sqrt{2}}$
Circle radius r	(1) Through centre, perpendicular to plane (i.e. polar axis)	$\frac{\pi r^4}{2}$	$\frac{r}{\sqrt{2}}$
	(2) Coinciding with diameter	$\frac{\pi r^4}{4}$	$\frac{r}{2}$
	(3) About a tangent	$\frac{5\pi r^4}{4}$	$\frac{\sqrt{5}}{2}r$
Semicircle radius r	Coinciding with diameter	$\frac{\pi r^4}{8}$	$\frac{r}{2}$

Solved problems on second moments of area of regular sections

Problem 1. Determine the second moment of area and the radius of gyration about axes AA, BB and CC for the rectangle shown in Fig. 39

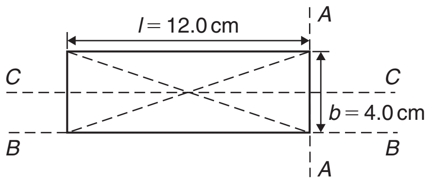

Figure 39

From Table 3, the second moment of area about axis AA, $\boldsymbol{I_{AA}} = \dfrac{bl^3}{3} = \dfrac{(4.0)(12.0)^3}{3} = \mathbf{2304\,cm^4}$

Radius of gyration,

$$\boldsymbol{k_{AA}} = \frac{l}{\sqrt{3}} = \frac{12.0}{\sqrt{3}} = \mathbf{6.93\,cm}$$

Similarly, $\boldsymbol{I_{BB}} = \dfrac{lb^3}{3} = \dfrac{(12.0)(4.0)^3}{3} = \mathbf{256\,cm^4}$

and $\boldsymbol{k_{BB}} = \dfrac{b}{\sqrt{3}} = \dfrac{4.0}{\sqrt{3}} = \mathbf{2.31\,cm}$

The second moment of area about the centroid of a rectangle is $\dfrac{bl^3}{12}$ when the axis through the centroid is parallel with the breadth, b. In this case, the axis CC is parallel with the length l.

Hence $\boldsymbol{I_{CC}} = \dfrac{lb^3}{12} = \dfrac{(12.0)(4.0)^3}{12} = \mathbf{64\,cm^4}$

and $\boldsymbol{k_{CC}} = \dfrac{b}{\sqrt{12}} = \dfrac{4.0}{\sqrt{12}} = \mathbf{1.15\,cm}$

Problem 2. Find the second moment of area and the radius of gyration about axis PP for the rectangle shown in Fig. 40

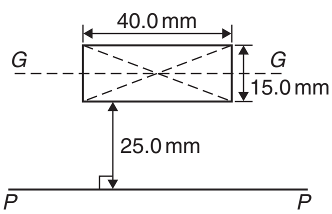

Figure 40

$I_{GG} = \dfrac{lb^3}{12}$ where $l = 40.0\,\text{mm}$ and $b = 15.0\,\text{mm}$

Hence $I_{GG} = \dfrac{(40.0)(15.0)^3}{12} = 11\,250\,\text{mm}^4$

From the parallel axis theorem, $I_{PP} = I_{GG} + Ad^2$, where $A = 40.0 \times 15.0 = 600\,\text{mm}^2$ and $d = 25.0 + 7.5 = 32.5\,\text{mm}$, the perpendicular distance between GG and PP.

Hence,
$$\boldsymbol{I_{PP}} = 11\,250 + (600)(32.5)^2 = \mathbf{645\,000\,mm^4}$$

$$I_{PP} = Ak_{PP}^2$$

from which,
$$\boldsymbol{k_{PP}} = \sqrt{\frac{I_{PP}}{\text{area}}} = \sqrt{\frac{645\,000}{600}} = \mathbf{32.79\,mm}$$

Problem 3. Determine the second moment of area and radius of gyration about axis QQ of the triangle BCD shown in Fig. 41

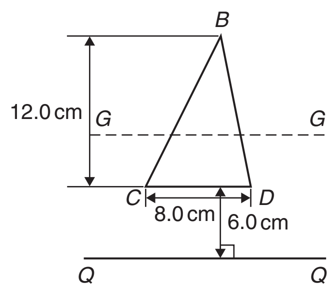

Figure 41

Using the parallel axis theorem: $I_{QQ} = I_{GG} + Ad^2$, where I_{GG} is the second moment of area about the centroid of the triangle,

i.e. $\dfrac{bh^3}{36} = \dfrac{(8.0)(12.0)^3}{36} = 384\,\text{cm}^4$, A is the area of the triangle $= \frac{1}{2}bh = \frac{1}{2}(8.0)(12.0) = 48\,\text{cm}^2$ and d is the distance between axes GG and $QQ = 6.0 + \frac{1}{3}(12.0) = 10\,\text{cm}$.

Hence the second moment of area about axis QQ,

$$\boldsymbol{I_{QQ}} = 384 + (48)(10)^2 = \mathbf{5184\,cm^4}$$

Radius of gyration,

$$\boldsymbol{k_{QQ}} = \sqrt{\frac{I_{QQ}}{\text{area}}} = \sqrt{\frac{5184}{48}} = \mathbf{10.4\,cm}$$

Problem 4. Determine the second moment of area and radius of gyration of the circle shown in Fig. 42 about axis YY

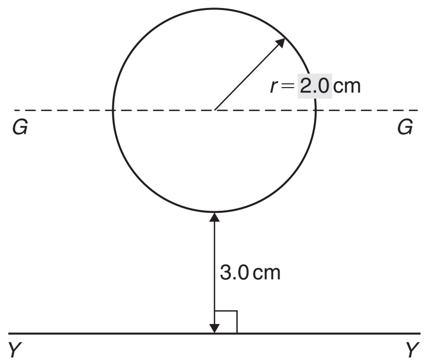

Figure 42

In Fig. 42, $I_{GG} = \dfrac{\pi r^4}{4} = \dfrac{\pi}{4}(2.0)^4 = 4\pi \text{ cm}^4$. Using the parallel axis theorem, $I_{YY} = I_{GG} + Ad^2$, where $d = 3.0 + 2.0 = 5.0$ cm.

$$\text{Hence} \quad \boldsymbol{I_{YY}} = 4\pi + [\pi(2.0)^2](5.0)^2$$

$$= 4\pi + 100\pi = 104\pi = \mathbf{327\ cm^4}$$

Radius of gyration,

$$\boldsymbol{k_{YY}} = \sqrt{\frac{I_{YY}}{\text{area}}} = \sqrt{\frac{104\pi}{\pi(2.0)^2}} = \sqrt{26} = \mathbf{5.10\ cm}$$

Problem 5. Determine the second moment of area and radius of gyration for the semicircle shown in Fig. 43 about axis XX

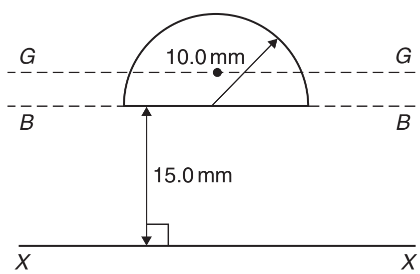

Figure 43

The centroid of a semicircle lies at $\dfrac{4r}{3\pi}$ from its diameter.

Using the parallel axis theorem: $I_{BB} = I_{GG} + Ad^2$,

$$\text{where} \quad I_{BB} = \frac{\pi r^4}{8} \text{(from Table 59.1)}$$

$$= \frac{\pi(10.0)^4}{8} = 3927 \text{ mm}^4,$$

$$A = \frac{\pi r^2}{2} = \frac{\pi(10.0)^2}{2} = 157.1 \text{ mm}^2$$

$$\text{and} \quad d = \frac{4r}{3\pi} = \frac{4(10.0)}{3\pi} = 4.244 \text{ mm}$$

$$\text{Hence} \quad 3927 = I_{GG} + (157.1)(4.244)^2$$

$$\text{i.e.} \quad 3927 = I_{GG} + 2830,$$

$$\text{from which,} \quad I_{GG} = 3927 - 2830 = 1097 \text{ mm}^4$$

Using the parallel axis theorem again:

$I_{XX} = I_{GG} + A(15.0 + 4.244)^2$

i.e. $\boldsymbol{I_{XX}} = 1097 + (157.1)(19.244)^2$

$= 1097 + 58\,179 = 59\,276 \text{ mm}^4$ or $\mathbf{59\,280\ mm^4}$, correct to 4 significant figures.

$$\text{Radius of gyration, } \boldsymbol{k_{XX}} = \sqrt{\frac{I_{XX}}{\text{area}}} = \sqrt{\frac{59\,276}{157.1}}$$

$$= \mathbf{19.42\ mm}$$

Problem 6. Determine the polar second moment of area of the propeller shaft cross-section shown in Fig. 44

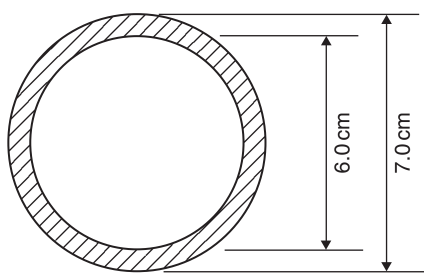

Figure 44

The polar second moment of area of a circle $= \dfrac{\pi r^4}{2}$. The polar second moment of area of the shaded area is given by the polar second moment of area of the 7.0 cm diameter circle minus the polar second moment of area of the 6.0 cm diameter circle. Hence the polar second moment of area of the

$$\text{cross-section shown} = \frac{\pi}{2}\left(\frac{7.0}{2}\right)^4 - \frac{\pi}{2}\left(\frac{6.0}{2}\right)^4$$

$$= 235.7 - 127.2 = \mathbf{108.5\,cm^4}$$

Problem 7. Determine the second moment of area and radius of gyration of a rectangular lamina of length 40 mm and width 15 mm about an axis through one corner, perpendicular to the plane of the lamina

The lamina is shown in Fig. 45.

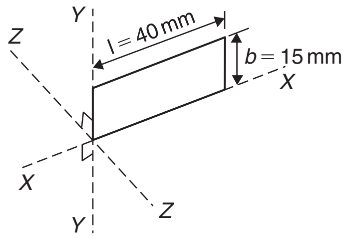

Figure 45

From the perpendicular axis theorem:

$$I_{ZZ} = I_{XX} + I_{YY}$$

$$I_{XX} = \frac{lb^3}{3} = \frac{(40)(15)^3}{3} = 45\,000\,\text{mm}^4$$

$$\text{and} \quad I_{YY} = \frac{bl^3}{3} = \frac{(15)(40)^3}{3} = 320\,000\,\text{mm}^4$$

$$\text{Hence} \quad \boldsymbol{I_{ZZ}} = 45\,000 + 320\,000$$

$$= \mathbf{365\,000\,mm^4} \text{ or } \mathbf{36.5\,cm^4}$$

Radius of gyration,

$$k_{ZZ} = \sqrt{\frac{I_{ZZ}}{\text{area}}} = \sqrt{\frac{365\,000}{(40)(15)}}$$

$$= \mathbf{24.7\,mm} \text{ or } \mathbf{2.47\,cm}$$

Exercise 22. Second moments of area of regular sections

Solved problems on second moments of area of composite areas

Problem 8. Determine correct to 3 significant figures, the second moment of area about XX for the composite area shown in Fig. 46.

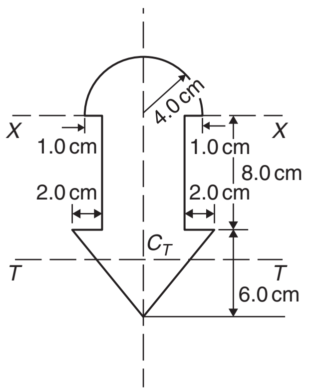

Figure 46

For the semicircle, $I_{XX} = \dfrac{\pi r^4}{8} = \dfrac{\pi(4.0)^4}{8}$

$= 100.5\,\text{cm}^4$

For the rectangle, $I_{XX} = \dfrac{bl^3}{3} = \dfrac{(6.0)(8.0)^3}{3}$

$= 1024\,\text{cm}^4$

For the triangle, about axis TT through centroid C_T,

$$I_{TT} = \frac{bh^3}{36} = \frac{(10)(6.0)^3}{36} = 60\,\text{cm}^4$$

By the parallel axis theorem, the second moment of area of the triangle about axis XX

$$= 60 + \left[\tfrac{1}{2}(10)(6.0)\right]\left[8.0 + \tfrac{1}{3}(6.0)\right]^2 = 3060\,\text{cm}^4.$$

Total second moment of area about XX.

$= 100.5 + 1024 + 3060 = 4184.5 = \mathbf{4180\,cm^4}$, correct to 3 significant figures

Problem 9. Determine the second moment of area and the radius of gyration about axis XX for the I-section shown in Fig. 47

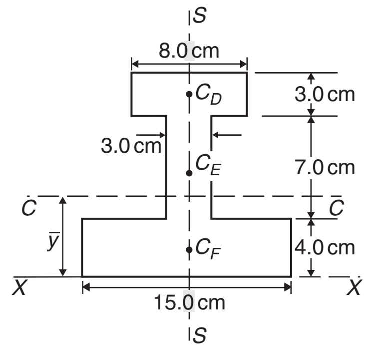

Figure 47

The I-section is divided into three rectangles, D, E and F and their centroids denoted by C_D, C_E and C_F respectively.

For rectangle D:
The second moment of area about C_D (an axis through C_D parallel to XX)

$$= \frac{bl^3}{12} = \frac{(8.0)(3.0)^3}{12} = 18\,\text{cm}^4$$

Using the parallel axis theorem: $I_{XX} = 18 + Ad^2$ where $A = (8.0)(3.0) = 24\,\text{cm}^2$ and $d = 12.5\,\text{cm}$

Hence $\boldsymbol{I_{XX}} = 18 + 24(12.5)^2 = \mathbf{3768\,cm^4}$

For rectangle E:
The second moment of area about C_E (an axis through C_E parallel to XX)

$$= \frac{bl^3}{12} = \frac{(3.0)(7.0)^3}{12} = 85.75\,\text{cm}^4$$

Using the parallel axis theorem:
$\boldsymbol{I_{XX}} = 85.75 + (7.0)(3.0)(7.5)^2 = \mathbf{1267\,cm^4}$

For rectangle F:

$$\boldsymbol{I_{XX}} = \frac{bl^3}{3} = \frac{(15.0)(4.0)^3}{3} = \mathbf{320\,cm^4}$$

Total second moment of area for the *I*-section about axis *XX*,

$$\boldsymbol{I_{XX}} = 3768 + 1267 + 320 = \mathbf{5355\,cm^4}$$

Total area of I-section

$$= (8.0)(3.0) + (3.0)(7.0) + (15.0)(4.0) = 105\,\text{cm}^2.$$

Radius of gyration,

$$\boldsymbol{k_{XX}} = \sqrt{\frac{I_{XX}}{\text{area}}} = \sqrt{\frac{5355}{105}} = \mathbf{7.14\,cm}$$

Exercise 23. Section moment of areas of composite areas