Module 14
Centroids of simple shapes

A. Centroids

A **lamina** is a thin flat sheet having uniform thickness. The **centre of gravity** of a lamina is the point where it balances perfectly, i.e. the lamina's **centre of mass**. When dealing with an area (i.e. a lamina of negligible thickness and mass) the term **centre of area** or **centroid** is used for the point where the centre of gravity of a lamina of that shape would lie.

The first moment of area

The **first moment of area** is defined as the product of the area and the perpendicular distance of its centroid from a given axis in the plane of the area. In Fig. 23, the first moment of area A about axis XX is given by (Ay) cubic units.

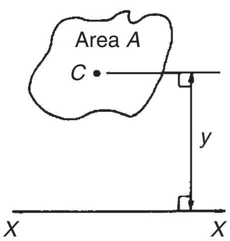

Figure 23

Centroid of area between a curve and the *x*-axis

(i) Figure 24 shows an area $PQRS$ bounded by the curve $y = f(x)$, the x-axis and ordinates $x = a$ and $x = b$. Let this area be divided into a large number of strips, each of width δx. A typical strip is shown shaded drawn at point (x, y) on $f(x)$. The area of the strip is approximately rectangular and is given by $y\delta x$. The centroid, C, has coordinates $\left(x, \frac{y}{2}\right)$.

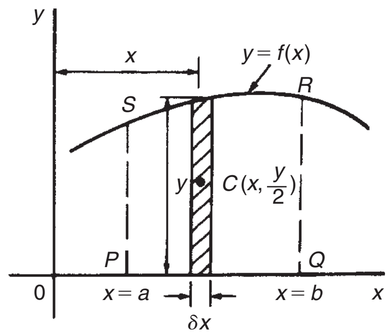

Figure 24

(ii) First moment of area of shaded strip about axis $Oy = (y\delta x)(x) = xy\delta x$.
Total first moment of area $PQRS$ about axis $Oy = \underset{\delta x \to 0}{\text{limit}} \sum_{x=a}^{x=b} xy\delta x = \int_a^b xy\,dx$

(iii) First moment of area of shaded strip about axis $Ox = (y\delta x)\left(\frac{y}{2}\right) = \frac{1}{2}y^2 x$.
Total first moment of area $PQRS$ about axis $Ox = \underset{\delta x \to 0}{\text{limit}} \sum_{x=a}^{x=b} \frac{1}{2}y^2\delta x = \frac{1}{2}\int_a^b y^2\,dx$

(iv) Area of $PQRS$, $A = \int_a^b y\,dx$

(v) Let $\bar{x}$ and $\bar{y}$ be the distances of the centroid of area A about Oy and Ox respectively then: $(\bar{x})(A) =$ total first moment of area A about axis $Oy = \int_a^b xy\,dx$

from which, $$\bar{x} = \frac{\int_a^b xy\,dx}{\int_a^b y\,dx}$$

and $(\bar{y})(A) =$ total moment of area A about axis $Ox = \frac{1}{2}\int_a^b y^2\,dx$

from which, $$\bar{y} = \frac{\frac{1}{2}\int_a^b y^2\,dx}{\int_a^b y\,dx}$$

Centroid of area between a curve and the y-axis

If $\bar{x}$ and $\bar{y}$ are the distances of the centroid of area $EFGH$ in Fig. 25 from Oy and Ox respectively, then, by similar reasoning as above:

$$(\bar{x})(\text{total area}) = \lim_{\delta y \to 0} \sum_{y=c}^{y=d} x\,\delta y\left(\frac{x}{2}\right) = \frac{1}{2}\int_c^d x^2\,dy$$

from which, $$\bar{x} = \frac{\frac{1}{2}\int_c^d x^2\,dy}{\int_c^d x\,dy}$$

and $$(\bar{y})(\text{total area}) = \lim_{\delta y \to 0} \sum_{y=c}^{y=d} (x\,\delta y)y = \int_c^d xy\,dy$$

from which, $$\bar{y} = \frac{\int_c^d xy\,dy}{\int_c^d x\,dy}$$

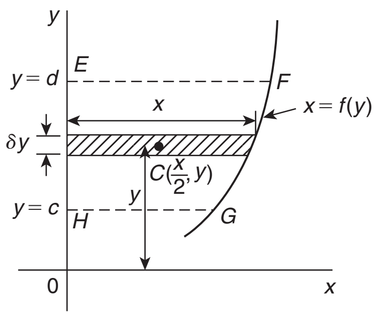

Figure 25

Solved problems on centroids of simple shapes

Problem 1. Show, by integration, that the centroid of a rectangle lies at the intersection of the diagonals

Let a rectangle be formed by the line $y=b$, the x-axis and ordinates $x=0$ and $x=l$ as shown in Fig. 26. Let the coordinates of the centroid C of this area be $(\bar{x}, \bar{y})$.

By integration, $$\bar{x} = \frac{\int_0^l xy\,dx}{\int_0^l y\,dx} = \frac{\int_0^l (x)(b)\,dx}{\int_0^l b\,dx}$$

$$= \frac{\left[b\frac{x^2}{2}\right]_0^l}{[bx]_0^l} = \frac{\frac{bl^2}{2}}{bl} = \frac{l}{2}$$

and $$\bar{y} = \frac{\frac{1}{2}\int_0^l y^2\,dx}{\int_0^l y\,dx} = \frac{\frac{1}{2}\int_0^l b^2\,dx}{bl}$$

$$= \frac{\frac{1}{2}[b^2x]_0^l}{bl} = \frac{\frac{b^2l}{2}}{bl} = \frac{b}{2}$$

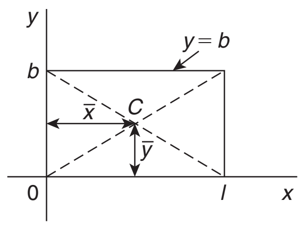

Figure 26

i.e. **the centroid lies at $\left(\frac{l}{2}, \frac{b}{2}\right)$ which is at the intersection of the diagonals.**

Problem 2. Find the position of the centroid of the area bounded by the curve $y = 3x^2$, the x-axis and the ordinates $x = 0$ and $x = 2$

If, $(\bar{x}, \bar{y})$ are the co-ordinates of the centroid of the given area then:

$$\bar{x} = \frac{\int_0^2 xy\,dx}{\int_0^2 y\,dx} = \frac{\int_0^2 x(3x^2)\,dx}{\int_0^2 3x^2\,dx}$$

$$= \frac{\int_0^2 3x^3\,dx}{\int_0^2 3x^2\,dx} = \frac{\left[\frac{3x^4}{4}\right]_0^2}{[x^3]_0^2} = \frac{12}{8} = \mathbf{1.5}$$

$$\bar{y} = \frac{\frac{1}{2}\int_0^2 y^2 dx}{\int_0^2 y dx} = \frac{\frac{1}{2}\int_0^2 (3x^2)^2 dx}{8}$$

$$= \frac{\frac{1}{2}\int_0^2 9x^4 dx}{8} = \frac{\frac{9}{2}\left[\frac{x^5}{5}\right]_0^2}{8} = \frac{\frac{9}{2}\left(\frac{32}{5}\right)}{8}$$

$$= \frac{18}{5} = \mathbf{3.6}$$

Hence the centroid lies at (1.5, 3.6)

Problem 3. Determine by integration the position of the centroid of the area enclosed by the line $y=4x$, the x-axis and ordinates $x=0$ and $x=3$

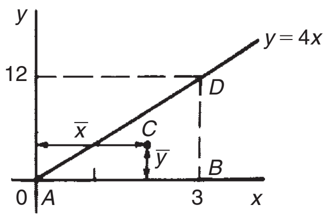

Figure 27

Let the coordinates of the area be $(\bar{x}, \bar{y})$ as shown in Fig. 27.

Then $$\bar{x} = \frac{\int_0^3 xy\,dx}{\int_0^3 y\,dx} = \frac{\int_0^3 (x)(4x)dx}{\int_0^3 4x\,dx}$$

$$= \frac{\int_0^3 4x^2 dx}{\int_0^3 4x\,dx} = \frac{\left[\frac{4x^3}{3}\right]_0^3}{[2x^2]_0^3} = \frac{36}{18} = 2$$

$$\bar{y} = \frac{\frac{1}{2}\int_0^3 y^2 dx}{\int_0^3 y\,dx} = \frac{\frac{1}{2}\int_0^3 (4x)^2 dx}{18}$$

$$= \frac{\frac{1}{2}\int_0^3 16x^2 dx}{18} = \frac{\frac{1}{2}\left[\frac{16x^3}{3}\right]_0^3}{18} = \frac{72}{18} = 4$$

Hence the centroid lies at (2, 4).

In Fig. 27, ABD is a right-angled triangle. The centroid lies 4 units from AB and 1 unit from BD showing that the centroid of a triangle lies at one-third of the perpendicular height above any side as base.

Exercise 19. Centroids of simple shapes

Solved problems on centroids of simple shapes

Problem 4. Determine the co-ordinates of the centroid of the area lying between the curve $y=5x-x^2$ and the x-axis

$y=5x-x^2=x(5-x)$. When $y=0$, $x=0$ or $x=5$, Hence the curve cuts the x-axis at 0 and 5 as shown in Fig. 28. Let the co-ordinates of the centroid be $(\bar{x}, \bar{y})$ then, by integration,

$$\bar{x} = \frac{\int_0^5 xy\,dx}{\int_0^5 y\,dx} = \frac{\int_0^5 x(5x-x^2)dx}{\int_0^5 (5x-x^2)dx}$$

$$= \frac{\int_0^5 (5x^2-x^3)dx}{\int_0^5 (5x-x^2)dx} = \frac{\left[\frac{5x^3}{3} - \frac{x^4}{4}\right]_0^5}{\left[\frac{5x^2}{2} - \frac{x^3}{3}\right]_0^5}$$

$$= \frac{\frac{625}{3} - \frac{625}{4}}{\frac{125}{2} - \frac{125}{3}} = \frac{\frac{625}{12}}{\frac{125}{6}}$$

$$= \left(\frac{625}{12}\right)\left(\frac{6}{125}\right) = \frac{5}{2} = \mathbf{2.5}$$

$$\bar{y} = \frac{\frac{1}{2}\int_0^5 y^2 dx}{\int_0^5 y\,dx} = \frac{\frac{1}{2}\int_0^5 (5x - x^2)^2 dx}{\int_0^5 (5x - x^2)\,dx}$$

$$= \frac{\frac{1}{2}\int_0^5 (25x^2 - 10x^3 + x^4)\ dx}{\frac{125}{6}}$$

$$= \frac{\frac{1}{2}\left[\frac{25x^3}{3} - \frac{10x^4}{4} + \frac{x^5}{5}\right]_0^5}{\frac{125}{6}}$$

$$= \frac{\frac{1}{2}\left(\frac{25(125)}{3} - \frac{6250}{4} + 625\right)}{\frac{125}{6}} = \mathbf{2.5}$$

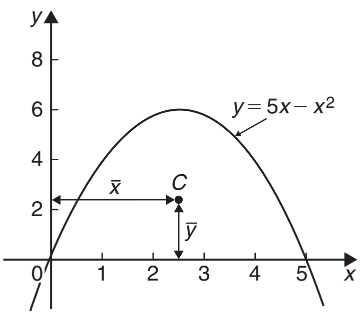

Figure 28

Hence the centroid of the area lies at (2.5, 2.5)(Note from Fig. 28 that the curve is symmetrical about $x = 2.5$ and thus $\bar{x}$ could have been determined 'on sight'.)

Problem 5. Locate the centroid of the area enclosed by the curve $y = 2x^2$, the y-axis and ordinates $y = 1$ and $y = 4$, correct to 3 decimal places

$$\bar{x} = \frac{\frac{1}{2}\int_1^4 x^2 dy}{\int_1^4 x\,dy} = \frac{\frac{1}{2}\int_1^4 \frac{y}{2} dy}{\int_1^4 \sqrt{\frac{y}{2}} dy}$$

$$= \frac{\frac{1}{2}\left[\frac{y^2}{4}\right]_1^4}{\left[\frac{2y^{3/2}}{3\sqrt{2}}\right]_1^4} = \frac{\frac{15}{8}}{\frac{14}{3\sqrt{2}}} = \mathbf{0.568}$$

and $$\bar{y} = \frac{\int_1^4 xy\,dy}{\int_1^4 x\,dy} = \frac{\int_1^4 \sqrt{\frac{y}{2}}(y)\,dy}{\frac{14}{3\sqrt{2}}}$$

$$= \frac{\int_1^4 \frac{y^{3/2}}{\sqrt{2}} dy}{\frac{14}{3\sqrt{2}}} = \frac{\frac{1}{\sqrt{2}}\left[\frac{y^{5/2}}{\frac{5}{2}}\right]_1^4}{\frac{14}{3\sqrt{2}}}$$

$$= \frac{\frac{2}{5\sqrt{2}}(31)}{\frac{14}{3\sqrt{2}}} = \mathbf{2.657}$$

Hence the position of the centroid is at (0.568, 2.657)

Problem 6. Locate the position of the centroid enclosed by the curves $y = x^2$ and $y^2 = 8x$

Figure 29 shows the two curves intersection at (0, 0) and (2, 4).

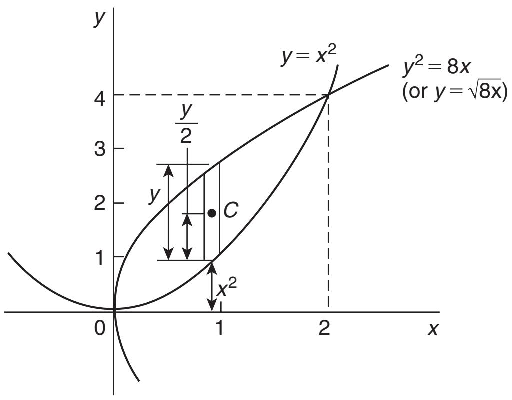

Figure 29

as $2\frac{2}{3}$ square units. Let the co-ordinates of centroid C be $\bar{x}$ and $\bar{y}$.

By integration, $\bar{x} = \dfrac{\int_0^2 xy\,dx}{\int_0^2 y\,dx}$

The value of y is given by the height of the typical strip shown in Fig. 29, i.e. $y = \sqrt{8x} - x^2$. Hence,

$$\bar{x} = \frac{\int_0^2 x(\sqrt{8x} - x^2)dx}{2\frac{2}{3}} = \frac{\int_0^2 (\sqrt{8}x^{3/2} - x^3)}{2\frac{2}{3}}$$

$$= \frac{\left[\sqrt{8}\frac{x^{5/2}}{\frac{5}{2}} - \frac{x^4}{4}\right]_0^2}{2\frac{2}{3}} = \left(\frac{\sqrt{8}\frac{\sqrt{2^5}}{\frac{5}{2}} - 4}{2\frac{2}{3}}\right)$$

$$= \frac{2\frac{2}{5}}{2\frac{2}{3}} = \mathbf{0.9}$$

Care needs to be taken when finding $\bar{y}$ in such examples as this. From Fig. 29, $y = \sqrt{8x} - x^2$ and $\frac{y}{2} = \frac{1}{2}(\sqrt{8x} - x^2)$. The perpendicular distance from centroid C of the strip to Ox is $\frac{1}{2}(\sqrt{8x} - x^2) + x^2$. Taking moments about Ox gives:

(total area) $(\bar{y}) = \sum_{x=0}^{x=2}$(area of strip) (perpendicular distance of centroid of strip to Ox)

Hence (area) $(\bar{y})$

$$= \int \left[\sqrt{8x} - x^2\right]\left[\frac{1}{2}(\sqrt{8x} - x^2) + x^2\right]dx$$

i.e. $\left(2\frac{2}{3}\right)$ $(\bar{y}) = \int_0^2 \left[\sqrt{8x} - x^2\right]\left(\frac{\sqrt{8x}}{2} + \frac{x^2}{2}\right)dx$

$$= \int_0^2 \left(\frac{8x}{2} - \frac{x^4}{2}\right)dx = \left[\frac{8x^2}{4} - \frac{x^5}{10}\right]_0^2$$

$$= \left(8 - 3\frac{1}{5}\right) - (0) = 4\frac{4}{5}$$

Hence $\quad \bar{y} = \dfrac{4\frac{4}{5}}{2\frac{2}{3}} = \mathbf{1.8}$

Thus the position of the centroid of the enclosed area in Fig. 29 is at (0.9, 1.8)

Exercise 20. Centroids of simple shapes

Theorem of Pappus

A theorem of Pappus states:
'If a plane area is rotated about an axis in its own plane but not intersecting it, the volume of the solid formed is given by the product of the area and the distance moved by the centroid of the area'.

With reference to Fig. 30, when the curve $y = f(x)$ is rotated one revolution about the x-axis between the limits $x = a$ and $x = b$, the volume V generated is given by:
volume $V = (A)(2\pi \bar{y})$, from which,

$$\bar{y} = \frac{V}{2\pi A}$$

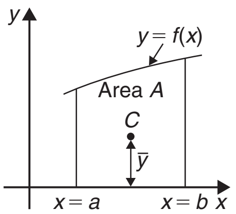

Figure 30

Problem 7. Determine the position of the centroid of a semicircle of radius r by using the theorem of Pappus. Check the answer by using integration (given that the equation of a circle, centre 0, radius r is $x^2 + y^2 = r^2$)

A semicircle is shown in Fig. 31 with its diameter lying on the x-axis and its centre at the origin. Area of semicircle $= \dfrac{\pi r^2}{2}$. When the area is rotated about the x-axis one revolution a sphere is generated of volume $\dfrac{4}{3}\pi r^3$.

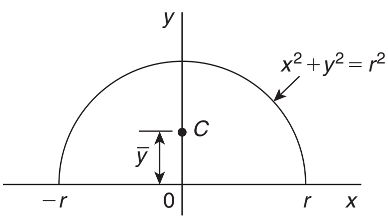

Figure 31

Let centroid C be at a distance $\bar{y}$ from the origin as shown in Fig. 31. From the theorem of Pappus,

volume generated = area × distance moved through by centroid i.e.

$$\frac{4}{3}\pi r^3 = \left(\frac{\pi r^2}{2}\right)(2\pi \bar{y})$$

Hence $\qquad \bar{y} = \dfrac{\frac{4}{3}\pi r^3}{\pi^2 r^2} = \dfrac{4r}{3\pi}$

By integration,

$$\bar{y} = \frac{\frac{1}{2}\int_{-r}^{r} y^2 dx}{\text{area}}$$

$$= \frac{\frac{1}{2}\int_{-r}^{r}(r^2 - x^2)dx}{\frac{\pi r^2}{2}} = \frac{\frac{1}{2}\left[r^2 x - \frac{x^3}{3}\right]_{-r}^{r}}{\frac{\pi r^2}{2}}$$

$$= \frac{\frac{1}{2}\left[\left(r^3 - \frac{r^3}{3}\right) - \left(-r^3 + \frac{r^3}{3}\right)\right]}{\frac{\pi r^2}{2}} = \frac{4r}{3\pi}$$

Hence the centroid of a semicircle lies on the axis of symmetry, distance $\dfrac{4r}{3\pi}$ (or $0.424\,r$) from its diameter.

Problem 8. Calculate the area bounded by the curve $y = 2x^2$, the x-axis and ordinates $x = 0$ and $x = 3$. (b) If this area is revolved (i) about the x-axis and (ii) about the y-axis, find the volumes of the solids produced. (c) Locate the position of the centroid using (i) integration, and (ii) the theorem of Pappus

(a) The required area is shown shaded in Fig. 32

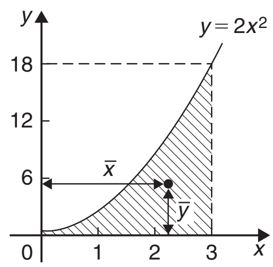

Figure 32

$$\text{Area} = \int_0^3 y\,dx = \int_0^3 2x^2\,dx = \left[\frac{2x^3}{3}\right]_0^3$$

$$= \mathbf{18\ square\ units}$$

(b) (i) When the shaded area of Fig. 32 is revolved 360° about the x-axis, the volume generated

$$= \int_0^3 \pi y^2 dx = \int_0^3 \pi (2x^2)^2 dx$$

$$= \int_0^3 4\pi x^4 dx = 4\pi \left[\frac{x^5}{5}\right]_0^3 = 4\pi\left(\frac{243}{5}\right)$$

$$= \mathbf{194.4\pi \text{ cubic units}}$$

(ii) When the shaded area of Fig. 32 is revolved 360° about the y-axis, the volume generated = (volume generated by $x = 3$) − (volume generated by $y = 2x^2$)

$$= \int_0^{18} \pi (3)^2 dy - \int_0^{18} \pi \left(\frac{y}{2}\right) dy$$

$$= \pi \int_0^{18} \left(9 - \frac{y}{2}\right) dy = \pi \left[9y - \frac{y^2}{4}\right]_0^{18}$$

$$= \mathbf{81\pi \text{ cubic units}}$$

(c) If the co-ordinates of the centroid of the shaded area in Fig. 32 are $(\bar{x}, \bar{y})$ then:

(i) by integration,

$$\bar{x} = \frac{\int_0^3 xy\,dx}{\int_0^3 y\,dx} = \frac{\int_0^3 x(2x^2)\,dx}{18}$$

$$= \frac{\int_0^3 2x^3 dx}{18} = \frac{\left[\frac{2x^4}{4}\right]_0^3}{18} = \frac{81}{36} = \mathbf{2.25}$$

$$\bar{y} = \frac{\frac{1}{2}\int_0^3 y^2 dx}{\int_0^3 y\,dx} = \frac{\frac{1}{2}\int_0^3 (2x^2)^2 dx}{18}$$

$$= \frac{\frac{1}{2}\int_0^3 4x^4 dx}{18} = \frac{\frac{1}{2}\left[\frac{4x^5}{5}\right]_0^3}{18} = \mathbf{5.4}$$

(ii) using the theorem of Pappus:
Volume generated when shaded area is revolved about Oy = (area)$(2\pi\bar{x})$

i.e. $\quad 81\pi = (18)(2\pi\bar{x})$,

from which, $\quad \bar{x} = \dfrac{81\pi}{36\pi} = \mathbf{2.25}$

Volume generated when shaded area is revolved about Ox = (area)$(2\pi\bar{y})$

i.e. $\quad 194.4\pi = (18)(2\pi\bar{y})$,

from which, $\quad \bar{y} = \dfrac{194.4\pi}{36\pi} = \mathbf{5.4}$

Hence the centroid of the shaded area in Fig. 32 is at (2.25, 5.4)

Problem 9. A cylindrical pillar of diameter 400 mm has a groove cut round its circumference. The section of the groove is a semicircle of diameter 50 mm. Determine the volume of material removed, in cubic centimetres, correct to 4 significant figures

A part of the pillar showing the groove is shown in Fig. 33.
The distance of the centroid of the semicircle from its base is $\dfrac{4r}{3\pi}$ (see Problem 7) $= \dfrac{4(25)}{3\pi} = \dfrac{100}{3\pi}$ mm.
The distance of the centroid from the centre of the pillar $= \left(200 - \dfrac{100}{3\pi}\right)$ mm.

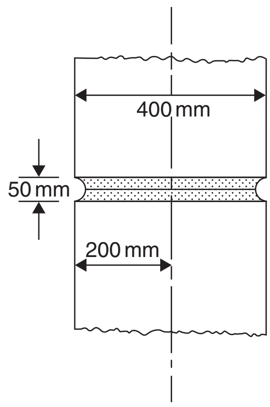

Figure 33

The distance moved by the centroid in one revolution

$$= 2\pi\left(200 - \frac{100}{3\pi}\right) = \left(400\pi - \frac{200}{3}\right) \text{mm.}$$

From the theorem of Pappus,
volume = area × distance moved by centroid

$$= \left(\frac{1}{2}\pi 25^2\right)\left(400\pi - \frac{200}{3}\right) = 1168250\,\text{mm}^3$$

Hence the volume of material removed is 1168 cm^3 correct to 4 significant figures.

Problem 10. A metal disc has a radius of 5.0 cm and is of thickness 2.0 cm. A semicircular groove of diameter 2.0 cm is machined centrally around the rim to form a pulley. Determine, using Pappus' theorem, the volume and mass of metal removed and the volume and mass of the pulley if the density of the metal is 8000 kg m^{-3}

A side view of the rim of the disc is shown in Fig. 34

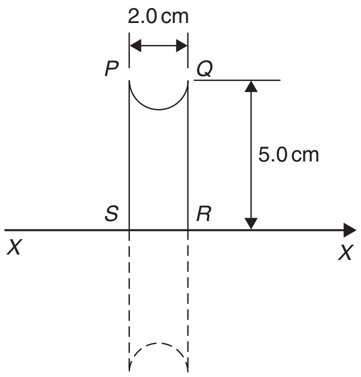

Figure 34

When area $PQRS$ is rotated about axis XX the volume generated is that of the pulley. The centroid of the semicircular area removed is at a distance of $\frac{4r}{3\pi}$ from its diameter (see Problem 7), i.e. $\frac{4(1.0)}{3\pi}$, i.e. 0.424 cm from PQ. Thus the distance of the centroid from XX is (5.0 − 0.424), i.e. 4.576 cm. The distance moved through in one revolution by the centroid is $2\pi(4.576)$ cm.
Area of semicircle

$$= \frac{\pi r^2}{2} = \frac{\pi(1.0)^2}{2} = \frac{\pi}{2}\,\text{cm}^2$$

By the theorem of Pappus, volume generated
= area × distance moved by centroid

$$= \left(\frac{\pi}{2}\right)(2\pi)(4.576)$$

i.e. **volume of metal removed = 45.16 cm^3**
Mass of metal removed = density × volume

$$= 8000\,\text{kg m}^{-3} \times \frac{45.16}{10^6}\,\text{m}^3$$
$$= \mathbf{0.3613\,kg} \text{ or } \mathbf{361.3\,g}$$

Volume of pulley = volume of cylindrical disc − volume of metal removed

$$= \pi(5.0)^2(2.0) - 45.16 = \mathbf{111.9\,cm^3}$$

Mass of pulley = density × volume

$$= 8000\,\text{kg m}^{-3} \times \frac{111.9}{10^6}\,\text{m}^3$$
$$= \mathbf{0.8952\,kg} \text{ or } \mathbf{895.2\,g}$$

Exercise 21. Theorem of Pappus