Module 13
Volumes of Solids of Revolution

A. Introduction

If the area under the curve $y = f(x)$, (shown in Fig. 15(a)), between $x = a$ and $x = b$ is rotated 360° about the x-axis, then a volume known as a **solid of revolution** is produced as shown in Fig. 15(b).

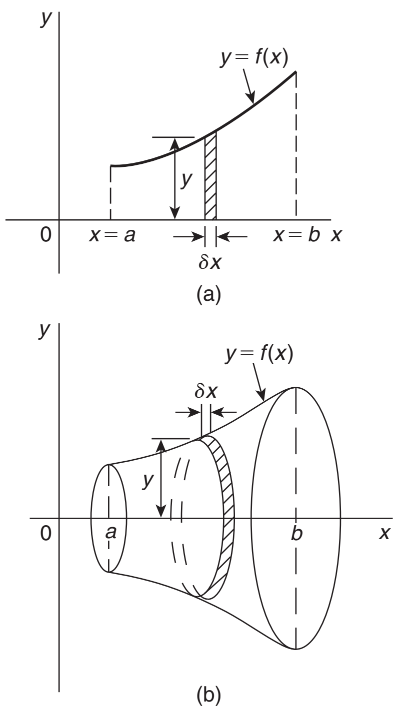

Figure 15

The volume of such a solid may be determined precisely using integration.

(i) Let the area shown in Fig. 15(a) be divided into a number of strips each of width δx. One such strip is shown shaded.

(ii) When the area is rotated 360° about the x-axis, each strip produces a solid of revolution approximating to a circular disc of radius y and thickness δx. Volume of disc = (circular cross-sectional area) (thickness) = $(\pi y^2)(\delta x)$

(iii) Total volume, V, between ordinates $x = a$ and $x = b$ is given by:

$$\textbf{Volume } V = \lim_{\delta x \to 0} \sum_{x=a}^{x=b} \pi y^2 \delta x = \int_a^b \pi y^2 \, dx$$

If a curve $x = f(y)$ is rotated about the y-axis 360° between the limits $y = c$ and $y = d$, as shown in Fig. 16, then the volume generated is given by:

$$\textbf{Volume } V = \lim_{\delta y \to 0} \sum_{y=c}^{y=d} \pi x^2 \delta y = \int_c^d \pi x^2 \, dy$$

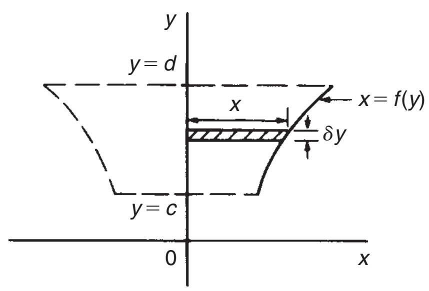

Figure 16

Solved problems on volumes of solids of revolution

Problem 1. Determine the volume of the solid of revolution formed when the curve $y = 2$ is rotated 360° about the x-axis between the limits $x = 0$ to $x = 3$

When $y = 2$ is rotated 360° about the x-axis between $x = 0$ and $x = 3$ (see Fig. 17):
volume generated

$$= \int_0^3 \pi y^2\, dx = \int_0^3 \pi (2)^2\, dx$$
$$= \int_0^3 4\pi\, dx = 4\pi [x]_0^3 = \mathbf{12\pi \text{ cubic units}}$$

[Check: The volume generated is a cylinder of radius 2 and height 3.
Volume of cylinder $= \pi r^2 h = \pi (2)^2 (3) = \mathbf{12\pi}$ **cubic units**.]

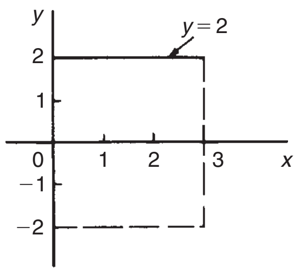

Figure 17

Problem 2. Find the volume of the solid of revolution when the cure $y = 2x$ is rotated one revolution about the x-axis between the limits $x = 0$ and $x = 5$

When $y = 2x$ is revolved one revolution about the x-axis between $x = 0$ and $x = 5$ (see Fig. 18) then:
volume generated

$$= \int_0^5 \pi y^2 dx = \int_0^5 \pi (2x)^2 dx$$
$$= \int_0^5 4\pi x^2 dx = 4\pi \left[\frac{x^3}{3}\right]_0^5$$
$$= \frac{500\pi}{3} = \mathbf{166\frac{2}{3}\pi \text{ cubic units}}$$

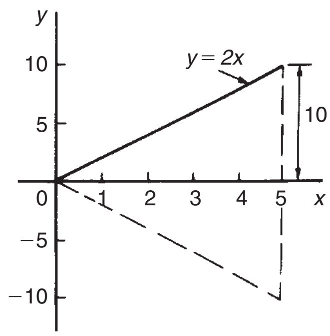

Figure 18

[Check: The volume generated is a cone of radius 10 and height 5. Volume of cone

$$= \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi (10)^2 5 = \frac{500\pi}{3}$$
$$= \mathbf{166\frac{2}{3}\pi \text{ cubic units}}.]$$

Problem 3. The curve $y = x^2 + 4$ is rotated one revolution about the x-axis between the limits $x = 1$ and $x = 4$. Determine the volume of the solid of revolution produced

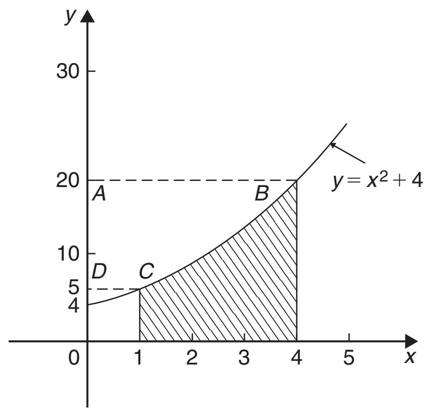

Figure 19

Revolving the shaded area shown in Fig. 19 about the x-axis 360° produces a solid of revolution given by:

$$\text{Volume} = \int_1^4 \pi y^2\, dx = \int_1^4 \pi (x^2 + 4)^2 dx$$
$$= \int_1^4 \pi (x^4 + 8x^2 + 16)\, dx$$

$$= \pi \left[\frac{x^5}{5} + \frac{8x^3}{3} + 16x \right]_1^4$$

$$= \pi[(204.8 + 170.67 + 64) - (0.2 + 2.67 + 16)]$$

$$= \mathbf{420.6\pi \text{ cubic units}}$$

Problem 4. If the curve in Problem 3 is revolved about the y-axis between the same limits, determine the volume of the solid of revolution produced

The volume produced when the curve $y = x^2 + 4$ is rotated about the y-axis between $y = 5$ (when $x = 1$) and $y = 20$ (when $x = 4$), i.e. rotating area ABCD of Fig. 19 about the y-axis is given by:

$$\text{volume} = \int_5^{20} \pi x^2 \, dy$$

Since $y = x^2 + 4$, then $x^2 = y - 4$

$$\text{Hence volume} = \int_5^{20} \pi(y-4)dy = \pi \left[\frac{y^2}{2} - 4y \right]_5^{20}$$

$$= \pi[(120) - (-7.5)]$$

$$= \mathbf{127.5\pi \text{ cubic units}}$$

Exercise 17. Volumes of solids of revolution

Soled problems on volumes of solids of revolution

Problem 5. The area enclosed by the curve $y = 3e^{\frac{x}{3}}$, the x-axis and ordinates $x = -1$ and $x = 3$ is rotated 360° about the x-axis. Determine the volume generated

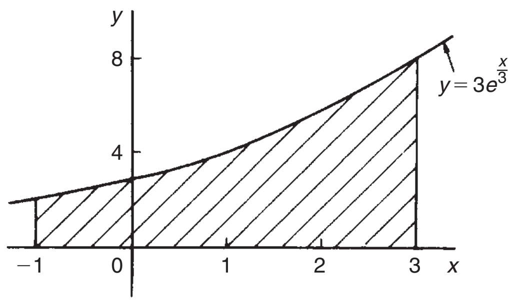

Figure 20

A sketch of $y = 3e^{\frac{x}{3}}$ is shown in Fig. 20. When the shaded area is rotated 360° about the x-axis then:

$$\text{volume generated} = \int_{-1}^{3} \pi y^2 \, dx$$

$$= \int_{-1}^{3} \pi \left(3e^{\frac{x}{3}}\right)^2 dx$$

$$= 9\pi \int_{-1}^{3} e^{\frac{2x}{3}} \, dx$$

$$= 9\pi\left[\frac{e^{\frac{2x}{3}}}{\frac{2}{3}}\right]_{-1}^{3}$$

$$= \frac{27\pi}{2}\left(e^2 - e^{-\frac{2}{3}}\right)$$

$$= \mathbf{92.82\pi \text{ cubic units}}$$

Problem 6. Determine the volume generated when the area above the x-axis bounded by the curve $x^2 + y^2 = 9$ and the ordinates $x = 3$ and $x = -3$ is rotated one revolution about the x-axis

Figure 21 shows the part of the curve $x^2 + y^2 = 9$ lying above the x-axis, Since, in general, $x^2 + y^2 = r^2$ represents a circle, centre 0 and radius r, then $x^2 + y^2 = 9$ represents a circle, centre 0 and radius 3. When the semi-circular area of Fig. 21 is rotated one revolution about the x-axis then:

$$\text{volume generated} = \int_{-3}^{3} \pi y^2 dx$$

$$= \int_{-3}^{3} \pi(9 - x^2)\,dx$$

$$= \pi\left[9x - \frac{x^3}{3}\right]_{-3}^{3}$$

$$= \pi[(18) - (-18)]$$

$$= \mathbf{36\pi \text{ cubic units}}$$

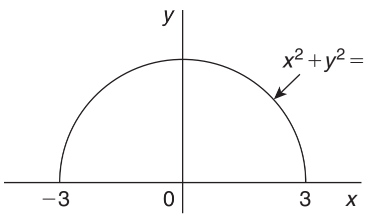

Figure 21

(Check: The volume generated is a sphere of radius 3. Volume of sphere $= \frac{4}{3}\pi r^3 = \frac{4}{3}\pi(3)^3 =$ **36π cubic units**.)

Problem 7. Calculate the volume of a frustum of a sphere of radius 4 cm that lies between two parallel planes at 1 cm and 3 cm from the centre and on the same side of it

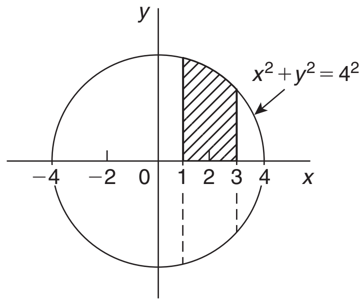

Figure 22

The volume of a frustum of a sphere may be determined by integration by rotating the curve $x^2 + y^2 = 4^2$ (i.e. a circle, centre 0, radius 4) one revolution about the x-axis, between the limits $x = 1$ and $x = 3$ (i.e. rotating the shaded area of Fig. 22).

$$\text{Volume of frustum} = \int_{1}^{3} \pi y^2\,dx$$

$$= \int_{1}^{3} \pi(4^2 - x^2)\,dx$$

$$= \pi\left[16x - \frac{x^3}{3}\right]_{1}^{3}$$

$$= \pi\left[(39) - \left(15\frac{2}{3}\right)\right]$$

$$= \mathbf{23\frac{1}{3}\pi \text{ cubic units}}$$

Problem 8. The area enclosed between the two parabolas $y = x^2$ and $y^2 = 8x$ is rotated 360° about the x-axis. Determine the volume of the solid produced

The volume produced by revolving the shaded area about the x-axis is given by: [(volume produced by revolving $y^2 = 8x$) − (volume produced by revolving $y = x^2$)]

i.e. $$\textbf{volume} = \int_{0}^{2} \pi(8x)\,dx - \int_{0}^{2} \pi(x^4)\,dx$$

$$= \pi\int_{0}^{2}(8x - x^4)\,dx = \pi\left[\frac{8x^2}{2} - \frac{x^5}{5}\right]_{0}^{2}$$

$$= \pi\left[\left(16 - \frac{32}{5}\right) - (0)\right]$$

$$= \mathbf{9.6\pi \text{ cubic units}}$$

Exercise 18. Volumes of solids of revolution