Module 12
Areas under & between curves

Area under a curve

The area shown shaded in Fig. 1 may be determined using approximate methods (such as the trapezoidal rule, the mid-ordinate rule or Simpson's rule) or, more precisely, by using integration.

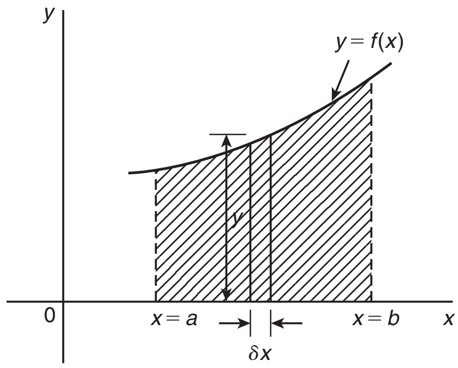

Figure 1

(i) Let A be the area shown shaded in Fig. 1 and let this area be divided into a number of strips each of width δx. One such strip is shown and let the area of this strip be δA.

Then: $\delta A \approx y\delta x$ (1)

The accuracy of statement (1) increases when the width of each strip is reduced, i.e. area A is divided into a greater number of strips.

(ii) Area A is equal to the sum of all the strips from $x = a$ to $x = b$,

i.e. $A = \lim_{\delta x \to 0} \sum_{x=a}^{x=b} y\,\delta x$ (2)

(iii) From statement (1), $\frac{\delta A}{\delta x} \approx y$ (3)

In the limit, as δx approaches zero, $\frac{\delta A}{\delta x}$ becomes the differential coefficient $\frac{dA}{dx}$

Hence $\lim_{\delta x \to 0}\left(\frac{\delta A}{\delta x}\right) = \frac{dA}{dx} = y$, from statement (3).

By integration,

$$\int \frac{dA}{dx}dx = \int y\,dx \quad \text{i.e.} \quad A = \int y\,dx$$

The ordinates $x = a$ and $x = b$ limit the area and such ordinate values are shown as limits. Hence

$$A = \int_a^b y\,dx \qquad (4)$$

(iv) Equating statements (2) and (4) gives:

$$\textbf{Area } A = \lim_{\delta x \to 0} \sum_{x=a}^{x=b} y\,\delta x = \int_a^b y\,dx$$

$$= \int_a^b f(x)\,dx$$

(v) If the area between a curve $x = f(y)$, the y-axis and ordinates $y = p$ and $y = q$ is required, then

$$\text{area} = \int_p^q x\,dy$$

Thus, determining the area under a curve by integration merely involves evaluating a definite integral.

There are several instances in engineering and science where the area beneath a curve needs to be accurately determined. For example, **the areas between limits of a**:

> **velocity/time graph gives distance travelled,**
> **force/distance graph gives work done,**
> **voltage/current graph gives power, and so on.**

Should a curve drop below the x-axis, then $y\ (=f(x))$ becomes negative and $f(x)\,dx$ is negative. When determining such areas by integration, a negative sign is placed before the integral. For the curve shown in Fig. 2, the total shaded area is given by (area $E+$ area $F+$area G).

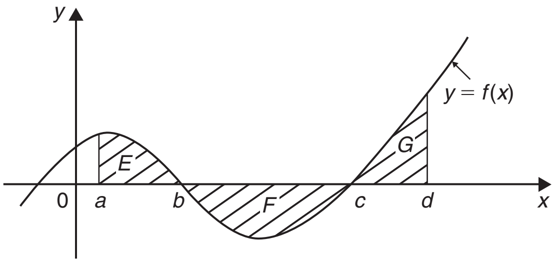

Figure 2

By integration, **total shaded area**

$$= \int_a^b f(x)\,dx - \int_b^c f(x)\,dx + \int_c^d f(x)\,dx$$

(Note that this is **not** the same as $\int_a^d f(x)\,dx$.)
It is usually necessary to sketch a curve in order to check whether it crosses the x-axis.

Solved problems on the area under a curve

Problem 1. Determine the area enclosed by $y=2x+3$, the x-axis and ordinates $x=1$ and $x=4$

$y=2x+3$ is a straight line graph as shown in Fig. 3, where the required area is shown shaded.
By integration,

$$\text{shaded area} = \int_1^4 y\,dx$$
$$= \int_1^4 (2x+3)\,dx$$
$$= \left[\frac{2x^2}{2}+3x\right]_1^4$$
$$= [(16+12)-(1+3)]$$
$$= \mathbf{24\ square\ units}$$

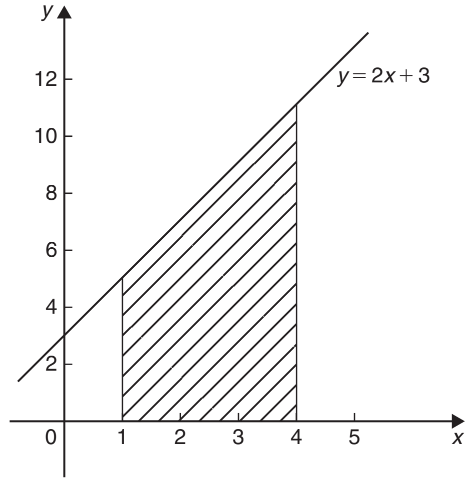

Figure 3

[This answer may be checked since the shaded area is a trapezium.

Area of trapezium

$$= \frac{1}{2}\binom{\text{sum of parallel}}{\text{sides}}\binom{\text{perpendicular distance}}{\text{between parallel sides}}$$
$$= \frac{1}{2}(5+11)(3)$$
$$= \mathbf{24\ square\ units}]$$

Problem 2. The velocity v of a body t seconds after a certain instant is: $(2t^2+5)$ m/s. Find by integration how far it moves in the interval from $t=0$ to $t=4$ s

Since $2t^2+5$ is a quadratic expression, the curve $v=2t^2+5$ is a parabola cutting the v-axis at $v=5$, as shown in Fig. 4.
The distance travelled is given by the area under the v/t curve (shown shaded in Fig. 4).
By integration,

$$\text{shaded area} = \int_0^4 v\,dt$$
$$= \int_0^4 (2t^2+5)\,dt$$
$$= \left[\frac{2t^3}{3}+5t\right]_0^4$$
$$= \left(\frac{2(4^3)}{3}+5(4)\right)-(0)$$

i.e. **distance travelled = 62.67 m**

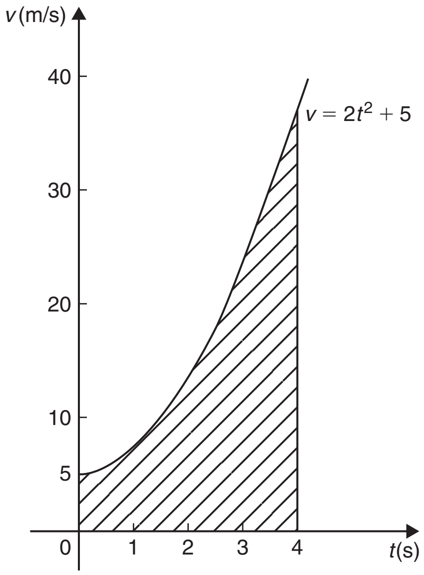

Figure 4

Problem 3. Sketch the graph $y = x^3 + 2x^2 - 5x - 6$ between $x = -3$ and $x = 2$ and determine the area enclosed by the curve and the x-axis

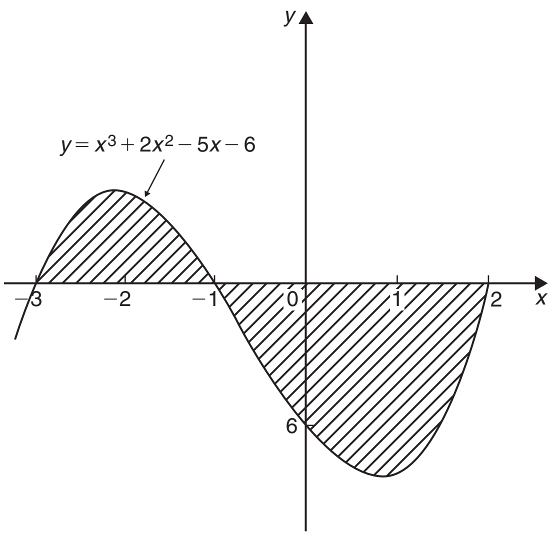

Figure 5

A table of values is produced and the graph sketched as shown in Fig. 5 where the area enclosed by the curve and the x-axis is shown shaded.

x	-3	-2	-1	0	1	2
x^3	-27	-8	-1	0	1	8
$2x^2$	18	8	2	0	2	8
$-5x$	15	10	5	0	-5	-10
-6	-6	-6	-6	-6	-6	-6
y	0	4	0	-6	-8	0

Shaded area $= \int_{-3}^{-1} y\,dx - \int_{-1}^{2} y\,dx$, the minus sign before the second integral being necessary since the enclosed area is below the x-axis.

Hence shaded area

$$= \int_{-3}^{-1} (x^3 + 2x^2 - 5x - 6)\,dx - \int_{-1}^{2} (x^3 + 2x^2 - 5x - 6)\,dx$$

$$= \left[\frac{x^4}{4} + \frac{2x^3}{3} - \frac{5x^2}{2} - 6x\right]_{-3}^{-1} - \left[\frac{x^4}{4} + \frac{2x^3}{3} - \frac{5x^2}{2} - 6x\right]_{-1}^{2}$$

$$= \left[\left\{\frac{1}{4} - \frac{2}{3} - \frac{5}{2} + 6\right\} - \left\{\frac{81}{4} - 18 - \frac{45}{2} + 18\right\}\right] - \left[\left\{4 + \frac{16}{3} - 10 - 12\right\} - \left\{\frac{1}{4} - \frac{2}{3} - \frac{5}{2} + 6\right\}\right]$$

$$= \left[\left\{3\frac{1}{12}\right\} - \left\{-2\frac{1}{4}\right\}\right] - \left[\left\{-12\frac{2}{3}\right\} - \left\{3\frac{1}{12}\right\}\right]$$

$$= \left[5\frac{1}{3}\right] - \left[-15\frac{3}{4}\right]$$

$$= \mathbf{21\frac{1}{12}} \text{ or } \mathbf{21.08 \text{ square units}}$$

Problem 4. Determine the area enclosed by the curve $y = 3x^2 + 4$, the x-axis and ordinates $x = 1$ and $x = 4$ by (a) the trapezoidal rule, (b) the mid-ordinate rule, (c) Simpson's rule, and (d) integration

x	0	1.0	1.5	2.0	2.5	3.0	3.5	4.0
y	4	7	10.75	16	22.75	31	40.75	52

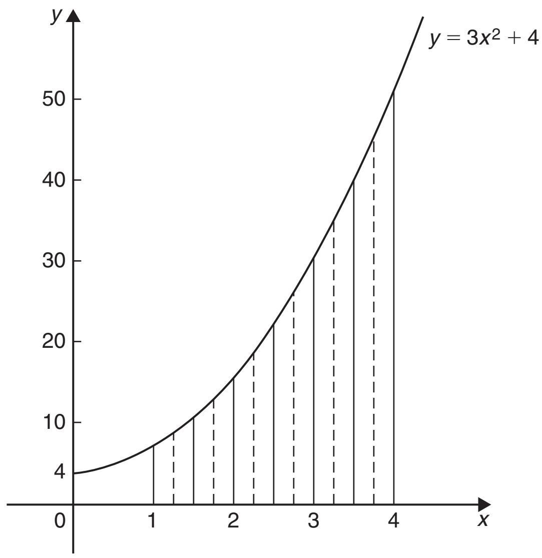

Figure 6

The curve $y=3x^2+4$ is shown plotted in Fig. 6.

(a) **By the trapezoidal rule**,

$$\text{Area} = \begin{pmatrix}\text{width of}\\ \text{interval}\end{pmatrix}\left[\frac{1}{2}\begin{pmatrix}\text{first}+\text{last}\\ \text{ordinate}\end{pmatrix}+\begin{pmatrix}\text{sum of}\\ \text{remaining}\\ \text{ordinates}\end{pmatrix}\right]$$

Selecting 6 intervals each of width 0.5 gives:

$$\text{Area} = (0.5)\left[\frac{1}{2}(7+52)+10.75+16 +22.75+31+40.75\right]$$

$$= \textbf{75.375 square units}$$

(b) **By the mid-ordinate rule**,
area = (width of interval) (sum of mid-ordinates). Selecting 6 intervals, each of width 0.5 gives the mid-ordinates as shown by the broken lines in Fig. 6.

$$\text{Thus, area} = (0.5)(8.5+13+19+26.5 +35.5+46)$$

$$= \textbf{74.25 square units}$$

(c) **By Simpson's rule**,

$$\text{area} = \frac{1}{3}\begin{pmatrix}\text{width of}\\ \text{interval}\end{pmatrix}\left[\begin{pmatrix}\text{first}+\text{last}\\ \text{ordinates}\end{pmatrix} +4\begin{pmatrix}\text{sum of even}\\ \text{ordinates}\end{pmatrix} +2\begin{pmatrix}\text{sum of remaining}\\ \text{odd ordinates}\end{pmatrix}\right]$$

Selecting 6 intervals, each of width 0.5, gives:

$$\text{area} = \frac{1}{3}(0.5)[(7+52)+4(10.75+22.75 +40.75)+2(16+31)]$$

$$= \textbf{75 square units}$$

(d) **By integration**, shaded area

$$= \int_1^4 y\,dx$$

$$= \int_1^4 (3x^2+4)\,dx$$

$$= \left[x^3+4x\right]_1^4$$

$$= \textbf{75 square units}$$

Integration gives the precise value for the area under a curve. In this case Simpson's rule is seen to be the most accurate of the three approximate methods.

Problem 5. Find the area enclosed by the curve $y = \sin 2x$, the x-axis and the ordinates $x=0$ and $x=\pi/3$

A sketch of $y=\sin 2x$ is shown in Fig. 7.

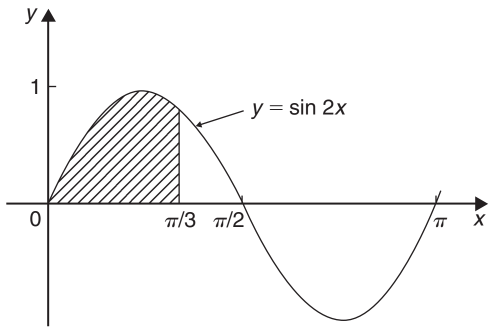

Figure 7

(Note that $y=\sin 2x$ has a period of $\frac{2\pi}{2}$, i.e. π radians.)

$$\text{Shaded area} = \int_0^{\pi/3} y\,dx$$
$$= \int_0^{\pi/3} \sin 2x\,dx$$
$$= \left[-\frac{1}{2}\cos 2x\right]_0^{\pi/3}$$
$$= \left\{-\frac{1}{2}\cos\frac{2\pi}{3}\right\} - \left\{-\frac{1}{2}\cos 0\right\}$$
$$= \left\{-\frac{1}{2}\left(-\frac{1}{2}\right)\right\} - \left\{-\frac{1}{2}(1)\right\}$$
$$= \frac{1}{4} + \frac{1}{2} = \mathbf{\frac{3}{4}\ \text{square units}}$$

Exercise 14. Area under curves

Solved problems on the area under a curve

Problem 6. A gas expands according to the law pv = constant. When the volume is $3\,m^3$ the pressure is 150 kPa. Given that work done $= \int_{v_1}^{v_2} p\,dv$, determine the work done as the gas expands from $2\,m^3$ to a volume of $6\,m^3$

pv = constant. When $v = 3\,m^3$ and $p = 150$ kPa the constant is given by $(3 \times 150) = 450\,kPa\,m^3$ or 450 kJ.

Hence $pv = 450$, or $p = \frac{450}{v}$

$$\text{Work done} = \int_2^6 \frac{450}{v}\,dv$$
$$= \Big[450 \ln v\Big]_2^6 = 450[\ln 6 - \ln 2]$$
$$= 450 \ln\frac{6}{2} = 450 \ln 3 = \mathbf{494.4\ kJ}$$

Problem 7. D eter mine the area enclosed by the curve $y = 4\cos\left(\frac{\theta}{2}\right)$, the θ-axis and ordinates $\theta = 0$ and $\theta = \frac{\pi}{2}$

The curve $y = 4\cos(\theta/2)$ is shown in Fig. 8.

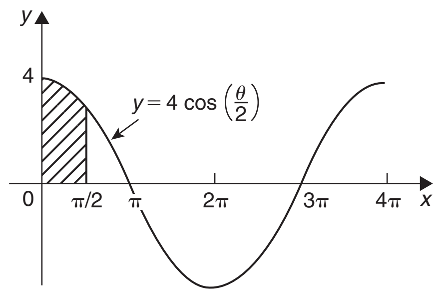

Figure 8

(Note that $y = 4\cos\left(\frac{\theta}{2}\right)$ has a maximum value of 4 and period $2\pi/(1/2)$, i.e. 4π rads.)

$$\text{Shaded area} = \int_0^{\pi/2} y\,d\theta = \int_0^{\pi/2} 4\cos\frac{\theta}{2}\,d\theta$$
$$= \left[4\left(\frac{1}{\frac{1}{2}}\right)\sin\frac{\theta}{2}\right]_0^{\pi/2}$$

$$= \left(8\sin\frac{\pi}{4}\right) - (8\sin 0)$$

$$= \mathbf{5.657\ square\ units}$$

Problem 8. Determine the area bounded by the curve $y = 3e^{t/4}$, the t-axis and ordinates $t = -1$ and $t = 4$, correct to 4 significant figures

A table of values is produced as shown.

t	-1	0	1	2	3	4
$y = 3e^{t/4}$	2.34	3.0	3.85	4.95	6.35	8.15

Since all the values of y are positive the area required is wholly above the t-axis.

$$\text{Hence area} = \int_1^4 y\,dt$$

$$= \int_1^4 3e^{t/4}dt = \left[\frac{3}{\left(\frac{1}{4}\right)}e^{t/4}\right]_{-1}^4$$

$$= 12\left[e^{t/4}\right]_{-1}^4 = 12(e^1 - e^{-1/4})$$

$$= 12(2.7183 - 0.7788)$$

$$= 12(1.9395) = \mathbf{23.27\ square\ units}$$

Problem 9. Sketch the curve $y = x^2 + 5$ between $x = -1$ and $x = 4$. Find the area enclosed by the curve, the x-axis and the ordinates $x = 0$ and $x = 3$. Determine also, by integration, the area enclosed by the curve and the y-axis, between the same limits

A table of values is produced and the curve $y = x^2 + 5$ plotted as shown in Fig. 9.

x	-1	0	1	2	3
y	6	5	6	9	14

$$\text{Shaded area} = \int_0^3 y\,dx = \int_0^3 (x^2 + 5)\,dx$$

$$= \left[\frac{x^3}{5} + 5x\right]_0^3$$

$$= \mathbf{24\ square\ units}$$

When $x = 3$, $y = 3^2 + 5 = 14$, and when $x = 0$, $y = 5$.

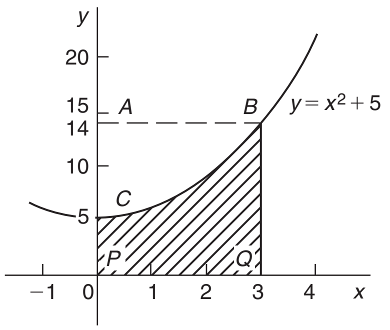

Figure 9

Since $y = x^2 + 5$ then $x^2 = y - 5$ and $x = \sqrt{y - 5}$

The area enclosed by the curve $y = x^2 + 5$ (i.e. $x = \sqrt{y - 5}$), the y-axis and the ordinates $y = 5$ and $y = 14$ (i.e. area ABC of Fig.) is given by:

$$\text{Area} = \int_{y=5}^{y=14} x\,dy = \int_5^{14} \sqrt{y - 5}\,dy$$

$$= \int_5^{14} (y - 5)^{1/2}\,dy$$

Let $u = y - 5$, then $\frac{du}{dy} = 1$ and $dy = du$

$$\text{Hence} \int (y - 5)^{1/2}dy = \int u^{1/2}du = \frac{2}{3}u^{3/2}$$

Since $u = y - 5$ then

$$\int_5^{14} \sqrt{y - 5}\,dy = \frac{2}{3}\left[(y - 5)^{3/2}\right]_5^{14}$$

$$= \frac{2}{3}[\sqrt{9^3} - 0]$$

$$= \mathbf{18\ square\ units}$$

(Check: From Fig. 9, area $BCPQ$ + area ABC = 24 + 18 = 42 square units, which is the area of rectangle $ABQP$.)

Problem 10. Determine the area between the curve $y = x^3 - 2x^2 - 8x$ and the x-axis

$$y = x^3 - 2x^2 - 8x = x(x^2 - 2x - 8)$$

$$= x(x + 2)(x - 4)$$

When $y = 0$, then $x = 0$ or $(x + 2) = 0$ or $(x - 4) = 0$, i.e. when $y = 0$, $x = 0$ or -2 or 4, which means that the curve crosses the x-axis at 0, -2 and 4. Since the curve is a continuous function, only one other co-ordinate value needs to be calculated before a sketch of the curve can be produced. When $x=1$, $y=-9$, showing that the part of the curve between $x=0$ and $x=4$ is negative. A sketch of $y=x^3-2x^2-8x$ is shown in Fig. 10. (Another method of sketching Fig. 10 would have been to draw up a table of values.)

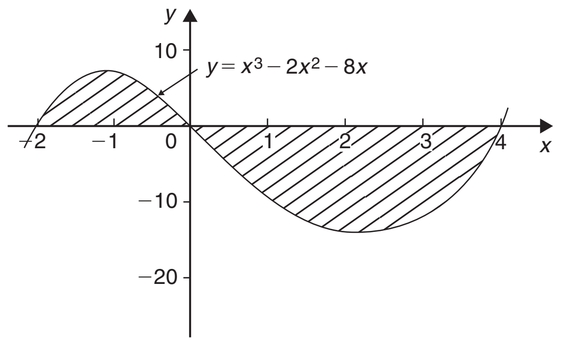

Figure 10

$$\text{Shaded area} = \int_{-2}^{0} (x^3 - 2x^2 - 8x)\, dx - \int_{0}^{4} (x^3 - 2x^2 - 8x)\, dx$$

$$= \left[\frac{x^4}{4} - \frac{2x^3}{3} - \frac{8x^2}{2}\right]_{-2}^{0} - \left[\frac{x^4}{4} - \frac{2x^3}{3} - \frac{8x^2}{2}\right]_{0}^{4}$$

$$= \left(6\frac{2}{3}\right) - \left(-42\frac{2}{3}\right)$$

$$= \mathbf{49\frac{1}{3}} \textbf{ square units}$$

Exercise 15. Areas under curves

Area between curves

The area enclosed between curves $y=f_1(x)$ and $y=f_2(x)$ (shown shaded in Fig. 11) is given by:

$$\text{shaded area} = \int_a^b f_2(x)\,dx - \int_a^b f_1(x)\,dx$$

$$= \int_a^b [f_2(x) - f_2(x)]\,dx$$

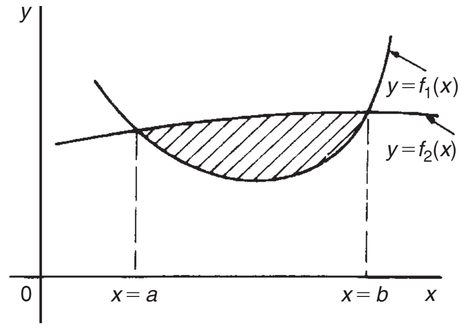

Figure 11

Problem 11. Determine the area enclosed between the curves $y=x^2+1$ and $y=7-x$

At the points of intersection, the curves are equal. Thus, equating the y-values of each curve gives: $x^2+1=7-x$, from which $x^2+x-6=0$. Factorising gives $(x-2)(x+3)=0$, from which, $x=2$ and $x=-3$. By firstly determining the points of intersection the range of x-values has been found. Tables of values are produced as shown below.

x	-3	-2	-1	0	1	2
$y=x^2+1$	10	5	2	1	2	5

x	-3	0	2
$y=7-x$	10	7	5

A sketch of the two curves is shown in Fig. 12.

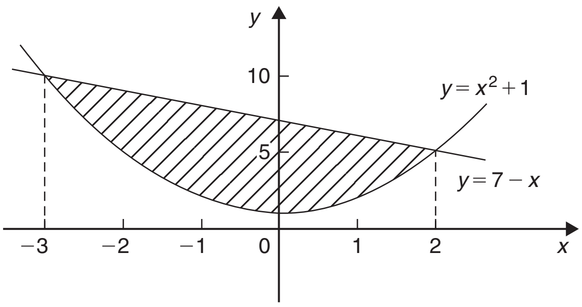

Figure 12

$$\text{Shaded area} = \int_{-3}^{2}(7-x)dx - \int_{-3}^{2}(x^2+1)dx$$

$$= \int_{-3}^{2}[(7-x)-(x^2+1)]dx$$

$$= \int_{-3}^{2}(6-x-x^2)dx$$

$$= \left[6x - \frac{x^2}{2} - \frac{x^3}{3}\right]_{-3}^{2}$$

$$= \left(12-2-\frac{8}{3}\right) - \left(-18-\frac{9}{2}+9\right)$$

$$= \left(7\frac{1}{3}\right) - \left(-13\frac{1}{2}\right)$$

$$= \mathbf{20\frac{5}{6}\ \text{square units}}$$

Problem 12. (a) Determine the coordinates of the points of intersection of the curves $y=x^2$ and $y^2=8x$. (b) Sketch the curves $y=x^2$ and $y^2=8x$ on the same axes. (c) Calculate the area enclosed by the two curves

(a) At the points of intersection the coordinates of the curves are equal. When $y=x^2$ then $y^2=x^4$.

Hence at the points of intersection $x^4=8x$, by equating the y^2 values.

Thus $x^4-8x=0$, from which $x(x^3-8)=0$, i.e. $x=0$ or $(x^3-8)=0$.

Hence at the points of intersection $x=0$ or $x=2$.

When $x=0$, $y=0$ and when $x=2$, $y=2^2=4$.

Hence the points of intersection of the curves $y=x^2$ and $y^2=8x$ are (0, 0) and (2, 4).

(b) A sketch of $y=x^2$ and $y^2=8x$ is shown in Fig. 13

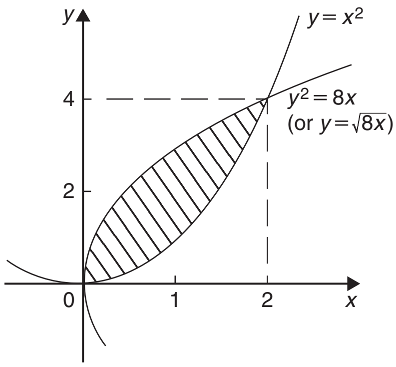

Figure 13

(c) $$\textbf{Shaded area} = \int_0^2 \{\sqrt{8x}-x^2\}dx$$

$$= \int_0^2 \{(\sqrt{8})x^{1/2}-x^2\}dx$$

$$= \left[(\sqrt{8})\frac{x^{3/2}}{\left(\frac{3}{2}\right)} - \frac{x^3}{3}\right]_0^2$$

$$= \left\{\frac{\sqrt{8}\sqrt{8}}{\left(\frac{3}{2}\right)} - \frac{8}{3}\right\} - \{0\}$$

$$= \frac{16}{3} - \frac{8}{3} = \frac{8}{3}$$

$$= \mathbf{2\frac{2}{3}\ \text{square units}}$$

Problem 13. Determine by integration the area bounded by the three straight lines $y = 4 - x$, $y = 3x$ and $3y = x$

Each of the straight lines is shown sketched in Fig. 14.

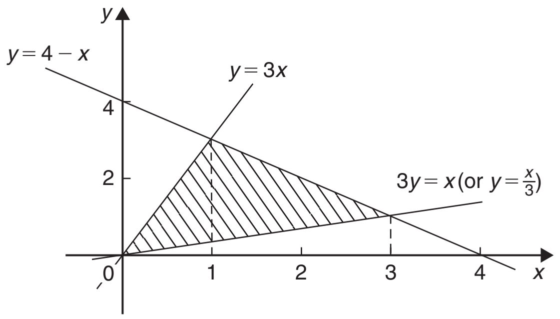

Figure 14

$$\text{Shaded area} = \int_0^1 \left(3x - \frac{x}{3}\right) dx + \int_1^3 \left[(4 - x) - \frac{x}{3}\right] dx$$

$$= \left[\frac{3x^2}{2} - \frac{x^2}{6}\right]_0^1 + \left[4x - \frac{x^2}{2} - \frac{x^2}{6}\right]_1^3$$

$$= \left[\left(\frac{3}{2} - \frac{1}{6}\right) - (0)\right]$$

$$+ \left[\left(12 - \frac{9}{2} - \frac{9}{6}\right) - \left(4 - \frac{1}{2} - \frac{1}{6}\right)\right]$$

$$= \left(1\frac{1}{3}\right) + \left(6 - 3\frac{1}{3}\right)$$

$$= \textbf{4 square units}$$

Exercise 16. Areas between curves