Module 11
Integration by parts

A. Introduction

From the product rule of differentiation:

$$\frac{d}{dx}(uv) = v\frac{du}{dx} + u\frac{dv}{dx}$$

where u and v are both functions of x.

Rearranging gives: $u\dfrac{dv}{dx} = \dfrac{d}{dx}(uv) - v\dfrac{du}{dx}$

Integrating both sides with respect to x gives:

$$\int u\frac{dv}{dx}dx = \int \frac{d}{dx}(uv)dx - \int v\frac{du}{dx}dx$$

ie $$\boldsymbol{\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx}$$

or $$\boldsymbol{\int u\,dv = uv - \int v\,du}$$

This is known as the **integration by parts formula** and provides a method of integrating such products of simple functions as $\int xe^x\,dx$, $\int t\,\sin t\,dt$, $\int e^\theta\cos\theta\,d\theta$ and $\int x\,\ln x\,dx$.

Given a product of two terms to integrate the initial choice is: 'which part to make equal to u' and 'which part to make equal to dv'. The choice must be such that the 'u part' becomes a constant after successive differentiation and the 'dv part' can be integrated from standard integrals. Invariable, the following rule holds: 'If a product to be integrated contains an algebraic term (such as x, t^2 or 3θ) then this term is chosen as the u part. The one exception to this rule is when a '$\ln x$' term is involved; in this case $\ln x$ is chosen as the 'u part'.

Solved problems on integration by parts

Problem 1. Determine: $\int x\cos x\,dx$

From the integration by parts formula,

$$\int u\,dv = uv - \int v\,du$$

Let $u = x$, from which $\dfrac{du}{dx} = 1$, i.e. $du = dx$ and let $dv = \cos x\,dx$, from which $v = \int\cos x\,dx = \sin x$.

Expressions for u, du and v are now substituted into the 'by parts' formula as shown below.

$$\begin{aligned} \int u \quad dv &= u \quad v - \int v \quad du \\ \int x \quad \cos x\,dx &= (x)\quad(\sin x) - \int (\sin x)\quad(dx) \end{aligned}$$

i.e. $$\int x\cos x\,dx = x\sin x - (-\cos x) + c$$

$$= \boldsymbol{x\sin x + \cos x + c}$$

[This result may be checked by differentiating the right hand side,

i.e. $\dfrac{d}{dx}(x\sin x + \cos x + c)$

$= [(x)(\cos x) + (\sin x)(1)] - \sin x + 0$ using the product rule

$= x\cos x$, which is the function being integrated.]

Problem 2. Find: $\int 3te^{2t}\,dt$

Let $u=3t$, from which, $\frac{du}{dt}=3$, i.e. $du=3\,dt$ and let $dv=e^{2t}\,dt$, from which, $v=\int e^{2t}\,dt=\frac{1}{2}e^{2t}$

Substituting into $\int u\,dv=u\,v-\int v\,du$ gives:

$$\int 3te^{2t}\,dt=(3t)\left(\frac{1}{2}e^{2t}\right)-\int\left(\frac{1}{2}e^{2t}\right)(3\,dt)$$
$$=\frac{3}{2}te^{2t}-\frac{3}{2}\int e^{2t}\,dt$$
$$=\frac{3}{2}te^{2t}-\frac{3}{2}\left(\frac{e^{2t}}{2}\right)+c$$

Hence $\displaystyle\int \mathbf{3te^{2t}\,dt=\frac{3}{2}e^{2t}\left(t-\frac{1}{2}\right)+c}$,

which may be checked by differentiating.

Problem 3. Evaluate: $\int_0^{\frac{\pi}{2}} 2\theta\sin\theta\,d\theta$

Let $u=2\theta$, from which, $\frac{du}{d\theta}=2$, i.e. $du=2\,d\theta$ and let $dv=\sin\theta\;d\theta$, from which,

$$v=\int\sin\theta\,d\theta=-\cos\theta$$

Substituting into $\int u\,dv=uv-\int v\,du$ gives:

$$\int 2\theta\sin\theta\,d\theta=(2\theta)(-\cos\theta)-\int(-\cos\theta)(2\,d\theta)$$
$$=-2\theta\cos\theta+2\int\cos\theta\,d\theta$$
$$=-2\theta\cos\theta+2\sin\theta+c$$

Hence $\int_0^{\frac{\pi}{2}} 2\theta\sin\theta\,d\theta$

$$=\Big[2\theta\cos\theta+2\sin\theta\Big]_0^{\frac{\pi}{2}}$$
$$=\left[-2\left(\frac{\pi}{2}\right)\cos\frac{\pi}{2}+2\sin\frac{\pi}{2}\right]-[0+2\sin 0]$$
$$=(-0+2)-(0+0)=\mathbf{2}$$

since $\cos\frac{\pi}{2}=0$ and $\sin\frac{\pi}{2}=1$

Problem 4. Evaluate: $\int_0^1 5xe^{4x}\,dx$, correct to 3 significant figures

Let $u=5x$, from which $\frac{du}{dx}=5$, i.e. $du=5\,dx$ and let $dv=e^{4x}\,dx$, from which, $v=\int e^{4x}dx=\frac{1}{4}e^{4x}$

Substituting into $\int u\,dv=uv-\int v\,du$ gives:

$$\int 5xe^{4x}\,dx=(5x)\left(\frac{e^{4x}}{4}\right)-\int\left(\frac{e^{4x}}{4}\right)(5\,dx)$$
$$=\frac{5}{4}xe^{4x}-\frac{5}{4}\int e^{4x}\,dx$$
$$=\frac{5}{4}xe^{4x}-\frac{5}{4}\left(\frac{e^{4x}}{4}\right)+c$$
$$=\frac{5}{4}e^{4x}\left(x-\frac{1}{4}\right)+c$$

Hence $\int_0^1 5xe^{4x}\,dx$

$$=\left[\frac{5}{4}e^{4x}\left(x-\frac{1}{4}\right)\right]_0^1$$
$$=\left[\frac{5}{4}e^4\left(1-\frac{1}{4}\right)\right]-\left[\frac{5}{4}e^0\left(0-\frac{1}{4}\right)\right]$$
$$=\left(\frac{15}{16}e^4\right)-\left(-\frac{5}{16}\right)$$
$$=51.186+0.313=51.499=\mathbf{51.5},$$

correct to 3 significant figures.

Problem 5. Determine: $\int x^2\sin x\,dx$

Let $u=x^2$, from which, $\frac{du}{dx}=2x$, i.e. $du=2x\,dx$, and let $dv=\sin x\,dx$, from which, $v=\int\sin x\,dx=-\cos x$

Substituting into $\int u\,dv=uv-\int v\,du$ gives:

$$\int x^2\sin x\,dx=(x^2)(-\cos x)-\int(-\cos x)(2x\,dx)$$
$$=-x^2\cos x+2\left[\int x\cos x\,dx\right]$$

The integral, $\int x\cos x\;dx$, is not a 'standard integral' and it can only be determined by using the integration by parts formula again.

From Problem 1, $\int x \cos x\, dx = x \sin x + \cos x$

Hence $\int x^2 \sin x\, dx$

$$= -x^2 \cos x + 2\{x \sin x + \cos x\} + c$$

$$= -x^2 \cos x + 2x \sin x + 2\cos x + c$$

$$= \mathbf{(2 - x^2) \cos x + 2x \sin x + c}$$

In general, if the algebraic term of a product is of power n, then the integration by parts formula is applied n times.

Exercise 12. Integration by parts

Solved problems on integration by parts

Problem 6. Find: $\int x \ln x\, dx$

The logarithmic function is chosen as the 'u part' Thus when $u = \ln x$, then $\frac{du}{dx} = \frac{1}{x}$ i.e. $du = \frac{dx}{x}$

Letting $dv = x\, dx$ gives $v = \int x\, dx = \frac{x^2}{2}$

Substituting into $\int u\, dv = uv - \int v\, du$ gives:

$$\int x \ln x\, dx = (\ln x)\left(\frac{x^2}{2}\right) - \int \left(\frac{x^2}{2}\right)\frac{dx}{x}$$

$$= \frac{x^2}{2} \ln x - \frac{1}{2}\int x\, dx$$

$$= \frac{x^2}{2} \ln x - \frac{1}{2}\left(\frac{x^2}{2}\right) + c$$

Hence $\int x \ln x\, dx = \mathbf{\frac{x^2}{2}\left(\ln x - \frac{1}{2}\right) + c}$

or $\mathbf{\frac{x^2}{4}(2 \ln x - 1) + c}$

Problem 7. Determine: $\int \ln x\, dx$

$\int \ln x\, dx$ is the same as $\int (1) \ln x\, dx$

Let $u = \ln x$, from which, $\frac{du}{dx} = \frac{1}{x}$ i.e. $du = \frac{dx}{x}$ and let $dv = 1\, dx$, from which, $v = \int 1\, dx = x$

Substituting into $\int u\, dv = uv - \int v\, du$ gives:

$$\int \ln x\, dx = (\ln x)(x) - \int x \frac{dx}{x}$$

$$= x \ln x - \int dx = x \ln x - x + c$$

Hence $\int \mathbf{\ln x\, dx = x(\ln x - 1) + c}$

Problem 8. Evaluate: $\int_1^9 \sqrt{x} \ln x\, dx$, correct to 3 significant figures

Let $u = \ln x$, from which $du = \dfrac{dx}{x}$
and let $dv = \sqrt{x}\,dx = x^{\frac{1}{2}}\,dx$, from which,

$$v = \int x^{\frac{1}{2}}\,dx = \frac{2}{3}x^{\frac{3}{2}}$$

Substituting into $\int u\,dv = uv - \int v\,du$ gives:

$$\begin{aligned}\int \sqrt{x}\ln x\,dx &= (\ln x)\left(\frac{2}{3}x^{\frac{3}{2}}\right) - \int\left(\frac{2}{3}x^{\frac{3}{2}}\right)\left(\frac{dx}{x}\right)\\ &= \frac{2}{3}\sqrt{x^3}\ln x - \frac{2}{3}\int x^{\frac{1}{2}}\,dx\\ &= \frac{2}{3}\sqrt{x^3}\ln x - \frac{2}{3}\left(\frac{2}{3}x^{\frac{3}{2}}\right) + c\\ &= \frac{2}{3}\sqrt{x^3}\left[\ln x - \frac{2}{3}\right] + c\end{aligned}$$

Hence $\displaystyle\int_1^9 \sqrt{x}\ln x\,dx = \left[\frac{2}{3}\sqrt{x^3}\left(\ln x - \frac{2}{3}\right)\right]_1^9$

$$\begin{aligned}&= \left[\frac{2}{3}\sqrt{9^3}\left(\ln 9 - \frac{2}{3}\right)\right] - \left[\frac{2}{3}\sqrt{1^3}\left(\ln 1 - \frac{2}{3}\right)\right]\\ &= \left[18\left(\ln 9 - \frac{2}{3}\right)\right] - \left[\frac{2}{3}\left(0 - \frac{2}{3}\right)\right]\\ &= 27.550 + 0.444 = 27.994 = \mathbf{28.0},\end{aligned}$$

correct to 3 significant figures.

Problem 9. Find: $\int e^{ax}\cos bx\,dx$

When integrating a product of an exponential and a sine or cosine function it is immaterial which part is made equal to 'u'.

Let $u = e^{ax}$, from which $\dfrac{du}{dx} = ae^{ax}$, i.e. $du = ae^{ax}\,dx$
and let $dv = \cos bx\,dx$, from which,

$$v = \int \cos bx\,dx = \frac{1}{b}\sin bx$$

Substituting into $\int u\,dv = uv - \int v\,du$ gives:

$$\begin{aligned}&\int e^{ax}\cos bx\,dx\\ &= (e^{ax})\left(\frac{1}{b}\sin bx\right) - \int\left(\frac{1}{b}\sin bx\right)(ae^{ax}dx)\\ &= \frac{1}{b}e^{ax}\sin bx - \frac{a}{b}\left[\int e^{ax}\sin bx\,dx\right] \qquad (1)\end{aligned}$$

$\int e^{ax}\sin bx\,dx$ is now determined separately using integration by parts again:

Let $u = e^{ax}$ then $du = ae^{ax}\,dx$, and let $dv = \sin bx\,dx$, from which

$$v = \int \sin bx\,dx = -\frac{1}{b}\cos bx$$

Substituting into the integration by parts formula gives:

$$\begin{aligned}\int e^{ax}\sin bx\,dx &= (e^{ax})\left(-\frac{1}{b}\cos bx\right)\\ &\qquad - \int\left(-\frac{1}{b}\cos bx\right)(ae^{ax}\,dx)\\ &= -\frac{1}{b}e^{ax}\cos bx\\ &\qquad + \frac{a}{b}\int e^{ax}\cos bx\,dx\end{aligned}$$

Substituting this result into equation (1) gives:

$$\begin{aligned}\int e^{ax}\cos bx\,dx &= \frac{1}{b}e^{ax}\sin bx - \frac{a}{b}\left[-\frac{1}{b}e^{ax}\cos bx\right.\\ &\qquad \left.+ \frac{a}{b}\int e^{ax}\cos bx\,dx\right]\\ &= \frac{1}{b}e^{ax}\sin bx + \frac{a}{b^2}e^{ax}\cos bx\\ &\qquad - \frac{a^2}{b^2}\int e^{ax}\cos bx\,dx\end{aligned}$$

The integral on the far right of this equation is the same as the integral on the left hand side and thus they may be combined.

$$\begin{aligned}\int e^{ax}\cos bx\,dx + \frac{a^2}{b^2}\int e^{ax}\cos bx\,dx\\ = \frac{1}{b}e^{ax}\sin bx + \frac{a}{b^2}e^{ax}\cos bx\end{aligned}$$

i.e. $$\left(1 + \frac{a^2}{b^2}\right)\int e^{ax}\cos bx\,dx = \frac{1}{b}e^{ax}\sin bx + \frac{a}{b^2}e^{ax}\cos bx$$

i.e. $$\left(\frac{b^2 + a^2}{b^2}\right)\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{b^2}(b\sin bx + a\cos bx)$$

Hence $\int e^{ax} \cos bx dx$

$$= \left(\frac{b^2}{b^2 + a^2}\right)\left(\frac{e^{ax}}{b^2}\right)(b \sin bx + a \cos bx)$$

$$= \frac{e^{ax}}{a^2 + b^2}(b \sin bx + a \cos bx) + c$$

Using a similar method to above, that is, integrating by parts twice, the following result may be proved:

$$\int e^{ax} \sin bx dx$$

$$= \frac{e^{ax}}{a^2 + b^2}(a \sin bx - b \cos bx) + c \qquad (2)$$

Problem 10. Evaluate: $\int_0^{\frac{\pi}{4}} e^t \sin 2t\, dt$, correct to 4 decimal places

Comparing $\int e^t \sin 2t\, dt$ with $\int e^{ax} \sin bx\, dx$ shows that $x = t$, $a = 1$ and $b = 2$.
Hence, substituting into equation (2) gives:

$$\int_0^{\frac{\pi}{4}} e^t \sin 2t\, dt$$

$$= \left[\frac{e^t}{1^2 + 2^2}(1 \sin 2t - 2 \cos 2t)\right]_0^{\frac{\pi}{4}}$$

$$= \left[\frac{e^{\frac{\pi}{4}}}{5}\left(\sin 2\left(\frac{\pi}{4}\right) - 2 \cos 2\left(\frac{\pi}{4}\right)\right)\right] - \left[\frac{e^0}{5}(\sin 0 - 2 \cos 0)\right]$$

$$= \left[\frac{e^{\frac{\pi}{4}}}{5}(1 - 0)\right] - \left[\frac{1}{5}(0 - 2)\right] = \frac{e^{\frac{\pi}{4}}}{5} + \frac{2}{5}$$

$= \mathbf{0.8387}$, correct to 4 decimal places

Exercise 13. Integration by parts