Module 10
Integration using Partial Fractions

A. Introduction

The process of expressing a fraction in terms of simpler fractions is called **partial fractions**

Certain functions have to be resolved into partial fractions before they an be integrated, as demonstrated in the following solved problems.

Solved problems on integration using partial fractions with linear factors

Problem 1. Determine: $\displaystyle\int \frac{11-3x}{x^2+2x-3}\,dx$

$$\frac{11-3x}{x^2+2x-3} \equiv \frac{2}{(x-1)} - \frac{5}{(x+3)}$$

$$\text{Hence } \int \frac{11-3x}{x^2+2x-3}dx$$

$$= \int \left\{\frac{2}{(x-1)} - \frac{5}{(x+3)}\right\} dx$$

$$= \mathbf{2\ ln(}\boldsymbol{x}\mathbf{-1) - 5\ ln(}\boldsymbol{x}\mathbf{+3) +} \boldsymbol{c}$$

(by algebraic substitutions)

$$\text{or } \mathbf{ln}\left\{\frac{(x-1)^2}{(x+3)^5}\right\} + c \text{ by the laws of logarithms}$$

Problem 2. Find: $\displaystyle\int \frac{2x^2-9x-35}{(x+1)(x-2)(x+3)}\,dx$

$$\frac{2x^2-9x-35}{(x+1)(x-2)(x+3)} \equiv \frac{4}{(x+1)} - \frac{3}{(x-2)} + \frac{1}{(x+3)}$$

$$\text{Hence } \int \frac{2x^2-9x-35}{(x+1)(x-2)(x+3)}dx$$

$$\equiv \int \left\{\frac{4}{(x+1)} - \frac{3}{(x-2)} + \frac{1}{(x+3)}\right\} dx$$

$$= \mathbf{4\ ln(}\boldsymbol{x}\mathbf{+1) - 3\ ln(}\boldsymbol{x}\mathbf{-2) + ln(}\boldsymbol{x}\mathbf{+3) +} \boldsymbol{c}$$

$$\text{or } \mathbf{ln}\left\{\frac{(x+1)^4(x+3)}{(x-2)^3}\right\} + c$$

Problem 3. Determine: $\displaystyle\int \frac{x^2+1}{x^2-3x+2}\,dx$

By dividing out (since the numerator and denominator are of the same degree) and resolving into partial fractions:

$$\frac{x^2+1}{x^2-3x+2} \equiv 1 - \frac{2}{(x-1)} + \frac{5}{(x-2)}$$

Hence $\int \frac{x^2+1}{x^2-3x+2}\,dx$

$$\equiv \int \left\{1 - \frac{2}{(x-1)} + \frac{5}{(x-2)}\right\} dx$$

$$= \mathbf{x - 2\ln(x-1) + 5\ln(x-2) + c}$$

$$\text{or } \mathbf{x + \ln\left\{\frac{(x-2)^5}{(x-1)^2}\right\} + c}$$

Problem 4. Evaluate: $\int_2^3 \frac{x^3-2x^2-4x-4}{x^2+x-2}\,dx$, correct to 4 significant figures

By dividing out and resolving into partial fractions,

$$\frac{x^3-2x^2-4x-4}{x^2+x-2} \equiv x - 3 + \frac{4}{(x+2)} - \frac{3}{(x-1)}$$

Hence $\int_2^3 \frac{x^3-2x^2-4x-4}{x^2+x-2}\,dx$

$$\equiv \int_2^3 \left\{x - 3 + \frac{4}{(x+2)} - \frac{3}{(x-1)}\right\} dx$$

$$= \left[\frac{x^2}{2} - 3x + 4\ln(x+2) - 3\ln(x-1)\right]_2^3$$

$$= \left(\frac{9}{2} - 9 + 4\ln 5 - 3\ln 2\right)$$

$$-(2 - 6 + 4\ln 4 - 3\ln 1)$$

$= \mathbf{-1.687}$, correct to 4 significant figures

Exercise 9. Integration using partial fractions with linear factors

Solved problems on integration using partial fractions with repeated linear factors

Problem 5. Determine: $\int \frac{2x+3}{(x-2)^2}\,dx$

$$\frac{2x+3}{(x-2)^2} \equiv \frac{2}{(x-2)} + \frac{7}{(x-2)^2}$$

Thus $\int \frac{2x+3}{(x-2)^2}\,dx$

$$\equiv \int \left\{ \frac{2}{(x-2)} + \frac{7}{(x-2)^2} \right\} dx$$

$$= \mathbf{2\ln(x-2) - \frac{7}{(x-2)} + c}$$

[$\int \frac{7}{(x-2)^2}\,dx$ is determined using the algebraic substitution $u \equiv (x-2)$]

Problem 6. Find: $\int \frac{5x^2-2x-19}{(x+3)(x-1)^2}\,dx$

$$\frac{5x^2-2x-19}{(x+3)(x-1)^2} \equiv \frac{2}{(x+3)} + \frac{3}{(x-1)} - \frac{4}{(x-1)^2}$$

Hence $\int \frac{5x^2-2x-19}{(x+3)(x-1)^2}\,dx$

$$\equiv \int \left\{ \frac{2}{(x+3)} + \frac{3}{(x-1)} - \frac{4}{(x-1)^2} \right\} dx$$

$$= \mathbf{2\ln(x+3) + 3\ln(x-1) + \frac{4}{(x-1)} + c}$$

$$\text{or} \quad \mathbf{\ln(x+3)^2(x-1)^3 + \frac{4}{(x-1)} + c}$$

Problem 7. Evaluate: $\int_{-2}^{1} \frac{3x^2+16x+15}{(x+3)^3}\,dx$, correct to 4 significant figures

$$\frac{3x^2+16x+15}{(x+3)^3} \equiv \frac{3}{(x+3)} - \frac{2}{(x+3)^2} - \frac{6}{(x+3)^3}$$

Hence $\int \frac{3x^2+16x+15}{(x+3)^3}\,dx$

$$\equiv \int_{-2}^{1} \left\{ \frac{3}{(x+3)} - \frac{2}{(x+3)^2} - \frac{6}{(x+3)^3} \right\} dx$$

$$= \left[3\ln(x+3) + \frac{2}{(x+3)} + \frac{3}{(x+3)^2} \right]_{-2}^{1}$$

$$= \left(3\ln 4 + \frac{2}{4} + \frac{3}{16} \right) - \left(3\ln 1 + \frac{2}{1} + \frac{3}{1} \right)$$

$= \mathbf{-0.1536}$, correct to 4 significant figures.

Exercise 10. Integration using partial fractions with repeated linear factors

Solved problems on integration using partial fractions with quadratic factors

Problem 8. Find: $\int \frac{3+6x+4x^2-2x^3}{x^2(x^2+3)}\,dx$

$$\frac{3+6x+4x^2-2x^2}{x^2(x^2+3)} \equiv \frac{2}{x}+\frac{1}{x^2}+\frac{3-4x}{(x^2+3)}$$

Thus $\int \frac{3+6x+4x^2-2x^3}{x^2(x^2+3)}\,dx$

$$\equiv \int\left(\frac{2}{x}+\frac{1}{x^2}+\frac{3-4x}{(x^2+3)}\right)dx$$

$$= \int\left\{\frac{2}{x}+\frac{1}{x^2}+\frac{3}{(x^2+3)}-\frac{4x}{(x^2+3)}\right\}dx$$

$$\int \frac{3}{(x^2+3)}\,dx = 3\int \frac{1}{x^2+(\sqrt{3})^2}\,dx$$

$$= \frac{3}{\sqrt{3}}\tan^{-1}\frac{x}{\sqrt{3}}$$

$\int \frac{4x}{x^2+3}\,dx$ is determined using the algebraic substitutions $u=(x^2+3)$.

Hence $\int\left\{\frac{2}{x}+\frac{1}{x^2}+\frac{3}{(x^2+3)}-\frac{4x}{(x^2+3)}\right\}dx$

$$= 2\ln x - \frac{1}{x} + \frac{3}{\sqrt{3}}\tan^{-1}\frac{x}{\sqrt{3}} - 2\ln(x^2+3)+c$$

$$= \mathbf{\ln\left(\frac{x}{x^2+3}\right)^2 - \frac{1}{x} + \sqrt{3}\tan^{-1}\frac{x}{\sqrt{3}} + c}$$

Problem 9. Determine: $\int \frac{1}{(x^2-a^2)}\,dx$

Let $\frac{1}{(x^2-a^2)} \equiv \frac{A}{(x-a)}+\frac{B}{(x+a)}$

$$\equiv \frac{A(x+a)+B(x-a)}{(x+a)(x-a)}$$

Equating the numerators gives:

$$1 \equiv A(x+a)+B(x-a)$$

Let $x=a$, then $A=\frac{1}{2a}$

and let $x=-a$,

then $B=-\frac{1}{2a}$

Hence $\int \frac{1}{(x^2-a^2)}\,dx \equiv \int \frac{1}{2a}\left[\frac{1}{(x-a)}-\frac{1}{(x+a)}\right]dx$

$$= \frac{1}{2a}[\ln(x-a)-\ln(x+a)]+c$$

$$= \mathbf{\frac{1}{2a}\ln\left(\frac{x-a}{x+a}\right)+c}$$

Problem 10. Evaluate: $\int_3^4 \frac{3}{(x^2-4)}\,dx$, correct to 3 significant figures

From Problem 9,

$$\int_3^4 \frac{3}{(x^2-4)}\,dx = 3\left[\frac{1}{2(2)}\ln\left(\frac{x-2}{x+2}\right)\right]_3^4$$

$$= \frac{3}{4}\left[\ln\frac{2}{6}-\ln\frac{1}{5}\right]$$

$$= \frac{3}{4}\ln\frac{5}{3} = \mathbf{0.383}, \text{ correct to 3 significant figures.}$$

Problem 11. Determine: $\int \frac{1}{(a^2-x^2)}\,dx$

Using partial fractions, let

$$\frac{1}{(a^2-x^2)} \equiv \frac{1}{(a-x)(a+x)} \equiv \frac{A}{(a-x)}+\frac{B}{(a+x)}$$

$$\equiv \frac{A(a+x)+B(a-x)}{(a-x)(a+x)}$$

Then $1 \equiv A(a+x)+B(a-x)$

Let $x=a$ then $A=\frac{1}{2a}$. Let $x=-a$ then $B=\frac{1}{2a}$

Hence $\int \frac{1}{(a^2 - x^2)} dx$

$$= \int \frac{1}{2a}\left[\frac{1}{(a-x)} + \frac{1}{(a+x)}\right] dx$$

$$= \frac{1}{2a}[-\ln(a-x) + \ln(a+x)] + c$$

$$= \mathbf{\frac{1}{2a}\ln\left(\frac{a+x}{a-x}\right) + c}$$

Problem 12. Evaluate: $\int_0^2 \frac{5}{(9-x^2)} dx$, correct to 4 decimal places

From Problem 11,

$$\int_0^2 \frac{5}{(9-x^2)} dx = 5\left[\frac{1}{2(3)}\ln\left(\frac{3+x}{3-x}\right)\right]_0^2$$

$$= \frac{5}{6}\left[\ln\frac{5}{1} - \ln 1\right] = \mathbf{1.3412},$$

correct to 4 decimal places

Exercise 11. Integration using partial fractions with quadratic factors