Module 1 - Differentiation Methods

A. Gradient of a Curve

If a tangent is drawn at a point P on a curve, then the gradient of this tangent is said to be the **gradient of the curve** at *P*. In Fig. 1, the gradient of the curve at *P* is equal to the gradient of the tangent *PQ*.

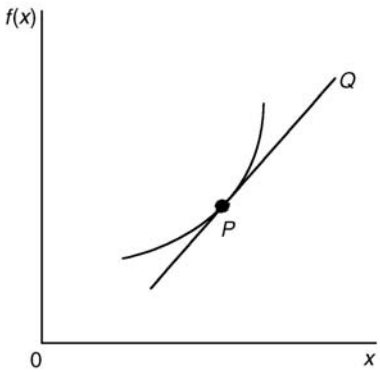

Figure 1

For the curve shown in Fig. 2, let the points *A* and *B* have co-ordinates (x_1, y_1) and (x_2, y_2), respectively. In functional notation, $y_1 = f(x_1)$ and $y_2 = f(x_2)$ as shown.

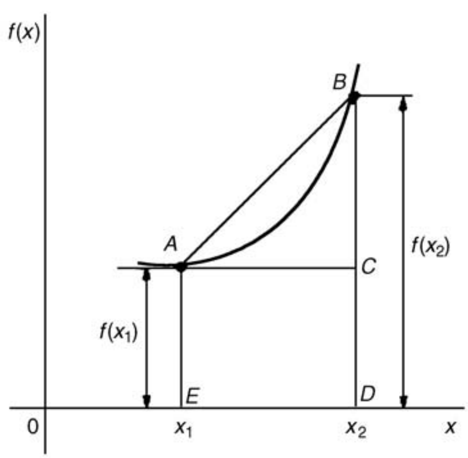

Figure 2

The gradient of the chord *AB*

$$= \frac{BC}{AC} = \frac{BD - CD}{ED} = \frac{f(x_2) - f(x_1)}{(x_2 - x_1)}$$

For the curve $f(x) = x^2$ shown in Fig. 3.

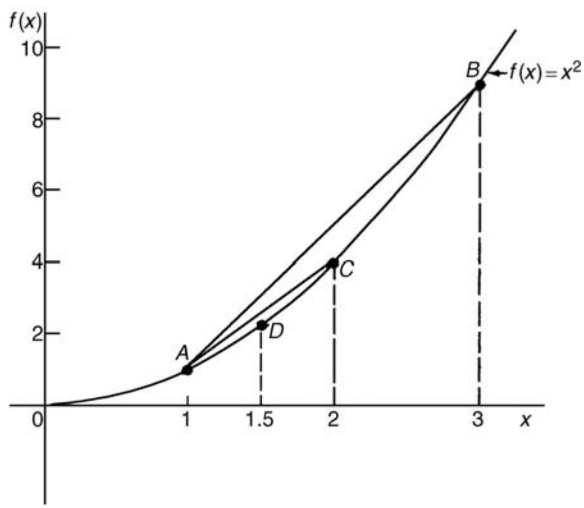

Figure 3

(i) the gradient of chord *AB*

$$= \frac{f(3) - f(1)}{3 - 1} = \frac{9 - 1}{2} = \mathbf{4}$$

(ii) the gradient of chord *AC*

$$= \frac{f(2) - f(1)}{2 - 1} = \frac{4 - 1}{1} = \mathbf{3}$$

(iii) the gradient of chord *AD*

$$= \frac{f(1.5) - f(1)}{1.5 - 1} = \frac{2.25 - 1}{0.5} = \mathbf{2.5}$$

(iv) if *E* is the point on the curve $(1.1,\ f(1.1))$ then the gradient of chord *AE*

$$= \frac{f(1.1) - f(1)}{1.1 - 1} = \frac{1.21 - 1}{0.1} = \mathbf{2.1}$$

(v) if F is the point on the curve $(1.01, f(1.01))$ then the gradient of chord AF

$$= \frac{f(1.01) - f(1)}{1.01 - 1} = \frac{1.0201 - 1}{0.01} = \mathbf{2.01}$$

Thus as point B moves closer and closer to point A the gradient of the chord approaches nearer and nearer to the value **2**. This is called the **limiting value** of the gradient of the chord AB and when B coincides with A the chord becomes the tangent to the curve.

B. Differentiation from first principles

In Fig. 4, A and B are two points very close together on a curve, δx (delta x) and δy (delta y) representing small increments in the x and y directions, respectively.

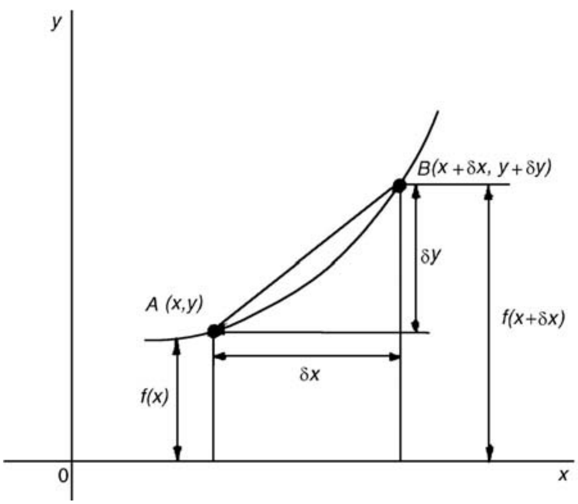

Figure 4

Gradient of chord $AB = \frac{\delta y}{\delta x}$; however, $\delta y = f(x + \delta x) - f(x)$.

Hence $\frac{\delta y}{\delta x} = \frac{f(x + \delta x) - f(x)}{\delta x}$.

As δx approaches zero, $\frac{\delta y}{\delta x}$ approaches a limiting value and the gradient of the chord approaches the gradient of the tangent at A.

When determining the gradient of a tangent to a curve there are two notations used. The gradient of the curve at A in Fig. 4 can either be written as

$$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} \quad \text{or} \quad \lim_{\delta x \to 0} \left\{ \frac{f(x + \delta x) - f(x)}{\delta x} \right\}$$

In **Leibniz notation**, $\mathbf{\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x}}$

In **functional notation**,

$$\mathbf{f'(x) = \lim_{\delta x \to 0} \left\{ \frac{f(x + \delta x) - f(x)}{\delta x} \right\}}$$

$\frac{dy}{dx}$ is the same as $f'(x)$ and is called the **differential coefficient** or the **derivative**. The process of finding the differential coefficient is called **differentiation**.

Problem 1. Differentiate from first principle $f(x) = x^2$ and determine the value of the gradient of the curve at $x = 2$.

To 'differentiate from first principles' means 'to find $f'(x)$' by using the expression

$$f'(x) = \lim_{\delta x \to 0} \left\{ \frac{f(x + \delta x) - f(x)}{\delta x} \right\}$$

$$f(x) = x^2$$

Substituting $(x + \delta x)$ for x gives $f(x + \delta x) = (x + \delta x)^2 = x^2 + 2x\delta x + \delta x^2$, hence

$$f'(x) = \lim_{\delta x \to 0} \left\{ \frac{(x^2 + 2x\delta x + \delta x^2) - (x^2)}{\delta x} \right\}$$

$$= \lim_{\delta x \to 0} \left\{ \frac{(2x\delta x + \delta x^2)}{\delta x} \right\}$$

$$= \lim_{\delta x \to 0} [2x + \delta x]$$

As $\delta x \to 0$, $[2x + \delta x] \to [2x + 0]$. Thus $\mathbf{f'(x) = 2x}$, i.e. the differential coefficient of x^2 is $2x$. At $x = 2$, the gradient of the curve, $f'(x) = 2(2) = \mathbf{4}$.

C. Differentiation of common functions

From differentiation by first principles of a number of examples such as in Problem 1 above, a general rule for differentiating $y = ax^n$ emerges, where a and n are constants.

The rule is: **if $y = ax^n$ then $\frac{dy}{dx} = anx^{n-1}$**

(or, **if $f(x) = ax^n$ then $f'(x) = anx^{n-1}$**) and is true for all real values of a and n.

For example, if $y = 4x^3$ then $a = 4$ and $n = 3$, and

$$\frac{dy}{dx} = anx^{n-1} = (4)(3)x^{3-1} = 12x^2$$

If $y = ax^n$ and $n = 0$ then $y = ax^0$ and

$$\frac{dy}{dx} = (a)(0)x^{0-1} = 0,$$

i.e. **the differential coefficient of a constant is zero**.

Figure 5(a) shows a graph of $y = \sin x$. The gradient is continually changing as the curve moves from 0 to A to B to C to D. The gradient, given by $\frac{dy}{dx}$, may be plotted in a corresponding position below $y = \sin x$, as shown in Fig. 5(b).

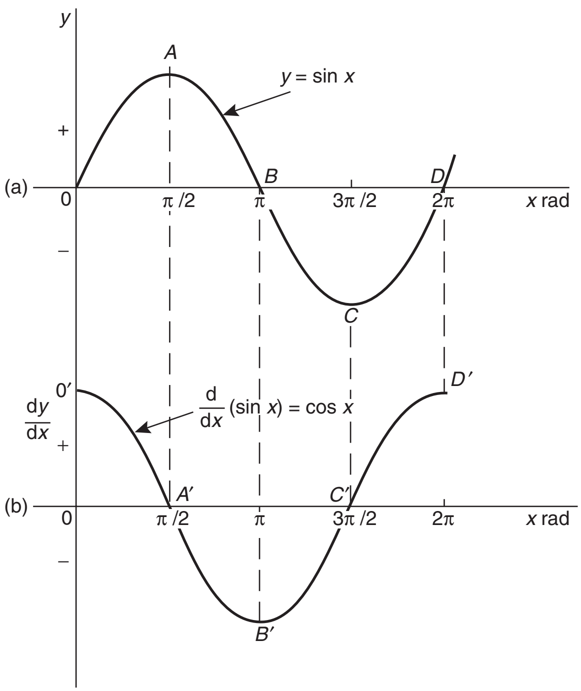

Figure 5

(i) At 0, the gradient is positive and is at its steepest. Hence 0′ is a maximum positive value.

(ii) Between 0 and A the gradient is positive but is decreasing in value until at A the gradient is zero, shown as A'.

(iii) Between A and B the gradient is negative but is increasing in value until at B the gradient is at its steepest negative value. Hence B' is a maximum negative value.

(iv) If the gradient of $y = \sin x$ is further investigated between B and D then the resulting graph of $\frac{dy}{dx}$ is seen to be a cosine wave. Hence the rate of change of $\sin x$ is $\cos x$,

i.e. **if $y = \sin x$ then $\frac{dy}{dx} = \cos x$**

By a similar construction to that shown in Fig. 5 it may be shown that:

$$\textbf{if } y = \sin ax \textbf{ then } \frac{dy}{dx} = a \cos ax$$

If graphs of $y = \cos x$, $y = e^x$ and $y = \ln x$ are plotted and their gradients investigated, their differential coefficients may be determined in a similar manner to that shown for $y = \sin x$. The rate of change of a function is a measure of the derivative.

The **standard derivatives** summarized below may be proved theoretically and are true for all real values of x

y or $f(x)$	$\frac{dy}{dx}$ or $f'(x)$
ax^n	anx^{n-1}
$\sin ax$	$a \cos ax$
$\cos ax$	$-a \sin ax$
e^{ax}	ae^{ax}
$\ln ax$	$\frac{1}{x}$

The **differential coefficient of a sum or difference** is the sum or difference of the differential coefficients of the separate terms.

Thus, if $f(x) = p(x) + q(x) - r(x)$,
(where f, p, q and r are functions),

then $f'(x) = p'(x) + q'(x) - r'(x)$

Differentiation of common functions is demonstrated in the following worked problems.

Problem 2. Find the differential coefficients of (a) $y = 12x^3$ (b) $y = \frac{12}{x^3}$.

If $y = ax^n$ then $\frac{dy}{dx} = anx^{n-1}$

(a) Since $y=12x^3$, $a=12$ and $n=3$ thus $\frac{dy}{dx}=(12)(3)x^{3-1}=\mathbf{36x^2}$

(b) $y=\frac{12}{x^3}$ is rewritten in the standard ax^n form as $y=12x^{-3}$ and in the general rule $a=12$ and $n=-3$.

Thus $\frac{dy}{dx}=(12)(-3)x^{-3-1}=-36x^{-4}=\mathbf{-\frac{36}{x^4}}$

Problem 3. Differentiate (a) $y=6$ (b) $y=6x$.

(a) $y=6$ may be written as $y=6x^0$, i.e. in the general rule $a=6$ and $n=0$.

Hence $\frac{dy}{dx}=(6)(0)x^{0-1}=\mathbf{0}$

In general, **the differential coefficient of a constant is always zero**.

(b) Since $y=6x$, in the general rule $a=6$ and $n=1$.

Hence $\frac{dy}{dx}=(6)(1)x^{1-1}=6x^0=\mathbf{6}$

In general, the differential coefficient of kx, where k is a constant, is always k.

Problem 4. Find the derivatives of (a) $y=3\sqrt{x}$ (b) $y=\frac{5}{\sqrt[3]{x^4}}$.

(a) $y=3\sqrt{x}$ is rewritten in the standard differential form as $y=3x^{\frac{1}{2}}$.

In the general rule, $a=3$ and $n=\frac{1}{2}$

$$\text{Thus } \frac{dy}{dx}=(3)\left(\frac{1}{2}\right)x^{\frac{1}{2}-1}=\frac{3}{2}x^{-\frac{1}{2}}$$

$$=\frac{3}{2x^{\frac{1}{2}}}=\mathbf{\frac{3}{2\sqrt{x}}}$$

(b) $y=\frac{5}{\sqrt[3]{x^4}}=\frac{5}{x^{\frac{4}{3}}}=5x^{-\frac{4}{3}}$ in the standard differential form.

In the general rule, $a=5$ and $n=-\frac{4}{3}$

$$\text{Thus } \frac{dy}{dx}=(5)\left(-\frac{4}{3}\right)x^{-\frac{4}{3}-1}=\frac{-20}{3}x^{-\frac{7}{3}}$$

$$=\frac{-20}{3x^{\frac{7}{3}}}=\mathbf{\frac{-20}{3\sqrt[3]{x^7}}}$$

Problem 5. Differentiate, with respect to x, $y=5x^4+4x-\frac{1}{2x^2}+\frac{1}{\sqrt{x}}-3$.

$y=5x^4+4x-\frac{1}{2x^2}+\frac{1}{\sqrt{x}}-3$ is rewritten as

$$y=5x^4+4x-\frac{1}{2}x^{-2}+x^{-\frac{1}{2}}-3$$

When differentiating a sum, each term is differentiated in turn.

$$\text{Thus } \frac{dy}{dx}=(5)(4)x^{4-1}+(4)(1)x^{1-1}-\frac{1}{2}(-2)x^{-2-1}+(1)\left(-\frac{1}{2}\right)x^{-\frac{1}{2}-1}-0$$

$$=20x^3+4+x^{-3}-\frac{1}{2}x^{-\frac{3}{2}}$$

$$\text{i.e. } \mathbf{\frac{dy}{dx}=20x^3+4+\frac{1}{x^3}-\frac{1}{2\sqrt{x^3}}}$$

Problem 6. Find the differential coefficients of (a) $y=3\sin 4x$ (b) $f(t)=2\cos 3t$ with respect to the variable.

(a) When $y=3\sin 4x$ then $\frac{dy}{dx}=(3)(4\cos 4x)$
$=\mathbf{12\cos 4x}$

(b) When $f(t)=2\cos 3t$ then
$f'(t)=(2)(-3\sin 3t)=\mathbf{-6\sin 3t}$

Problem 7. Determine the derivatives of (a) $y=3e^{5x}$ (b) $f(\theta)=\frac{2}{e^{3\theta}}$ (c) $y=6\ln 2x$.

(a) When $y=3e^{5x}$ then $\frac{dy}{dx}=(3)(5)e^{5x}=\mathbf{15e^{5x}}$

(b) $f(\theta)=\frac{2}{e^{3\theta}}=2e^{-3\theta}$, thus

$f'(\theta)=(2)(-3)e^{-30}=-6e^{-3\theta}=\mathbf{\frac{-6}{e^{3\theta}}}$

(c) When $y = 6 \ln 2x$ then $\frac{dy}{dx} = 6\left(\frac{1}{x}\right) = \mathbf{\frac{6}{x}}$

Problem 8. Find the gradient of the curve $y = 3x^4 - 2x^2 + 5x - 2$ at the points (0, −2) and (1, 4).

The gradient of a curve at a given point is given by the corresponding value of the derivative. Thus, since $y = 3x^4 - 2x^2 + 5x - 2$.

then the gradient $= \frac{dy}{dx} = 12x^3 - 4x + 5$.

At the point (0, −2), $x = 0$.
Thus the gradient $= 12(0)^3 - 4(0) + 5 = \mathbf{5}$.

At the point (1, 4), $x = 1$.
Thus the gradient $= 12(1)^3 - 4(1) + 5 = \mathbf{13}$.

Problem 9. Determine the co-ordinates of the point on the graph $y = 3x^2 - 7x + 2$ where the gradient is −1.

The gradient of the curve is given by the derivative.

When $y = 3x^2 - 7x + 2$ then $\frac{dy}{dx} = 6x - 7$

Since the gradient is −1 then $6x - 7 = -1$, from which, $x = 1$

When $x = 1$, $y = 3(1)^2 - 7(1) + 2 = -2$

Hence the gradient is −1 at the point (1, −2).

Exercise 1. Differentiation of common functions

D. Differentiation of a Product

When $y = uv$, and u and v are both functions of x,

then $$\frac{\mathbf{dy}}{\mathbf{dx}} = \boldsymbol{u}\frac{\mathbf{d}\boldsymbol{v}}{\mathbf{dx}} + \boldsymbol{v}\frac{\mathbf{d}\boldsymbol{u}}{\mathbf{dx}}$$

This is known as the **product rule**.

Problem 10. Find the differential coefficient of $y = 3x^2 \sin 2x$.

$3x^2 \sin 2x$ is a product of two terms $3x^2$ and $\sin 2x$
Let $u = 3x^2$ and $v = \sin 2x$
Using the product rule:

$$\frac{dy}{dx} = u \frac{dv}{dx} + v \frac{du}{dx}$$

gives: $$\frac{dy}{dx} = (3x^2)(2\cos 2x) + (\sin 2x)(6x)$$

i.e. $$\frac{dy}{dx} = 6x^2 \cos 2x + 6x \sin 2x$$
$$= \mathbf{6x(x\cos 2x + \sin 2x)}$$

Note that the differential coefficient of a product is **not** obtained by merely differentiating each term and multiplying the two answers together. The product rule formula **must** be used when differentiating products.

Problem 11. Find the rate of change of y with respect to x given $y = 3\sqrt{x} \ln 2x$.

The rate of change of y with respect to x is given by $\frac{dy}{dx}$

$y = 3\sqrt{x} \ln 2x = 3x^{\frac{1}{2}} \ln 2x$, which is a product.

Let $u = 3x^{\frac{1}{2}}$ and $v = \ln 2x$

Then $$\frac{dy}{dx} = u \frac{dv}{dx} + v \frac{du}{dx}$$
$$= \left(3x^{\frac{1}{2}}\right)\left(\frac{1}{x}\right) + (\ln 2x)\left[3\left(\frac{1}{2}\right)x^{\frac{1}{2}-1}\right]$$
$$= 3x^{\frac{1}{2}-1} + (\ln 2x)\left(\frac{3}{2}\right)x^{-\frac{1}{2}}$$
$$= 3x^{-\frac{1}{2}}\left(1 + \frac{1}{2}\ln 2x\right)$$

i.e. $$\frac{\mathbf{dy}}{\mathbf{dx}} = \frac{\mathbf{3}}{\sqrt{\boldsymbol{x}}}\left(\mathbf{1} + \frac{\mathbf{1}}{\mathbf{2}}\,\mathbf{ln\ 2}\boldsymbol{x}\right)$$

Problem 12. Differentiate $y = x^3 \cos 3x \ln x$.

Let $u = x^3 \cos 3x$ (i.e. a product) and $v = \ln x$

Then $$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

where $$\frac{du}{dx} = (x^3)(-3\sin 3x) + (\cos 3x)(3x^2)$$

and $$\frac{dv}{dx} = \frac{1}{x}$$

Hence $$\frac{dy}{dx} = (x^3\cos 3x)\left(\frac{1}{x}\right) + (\ln x)[-3x^3 \sin 3x + 3x^2 \cos 3x]$$
$$= x^2 \cos 3x + 3x^2 \ln x(\cos 3x - x \sin 3x)$$

i.e. $$\frac{\mathbf{dy}}{\mathbf{dx}} = \boldsymbol{x^2\{\cos 3x + 3\ln x(\cos 3x - x\sin 3x)\}}$$

Problem 13. Determine the rate of change of voltage, given $v = 5t \sin 2t$ volts when $t = 0.2$ s.

Rate of change of voltage $= \frac{dv}{dt}$
$$= (5t)(2\cos 2t) + (\sin 2t)(5)$$
$$= 10t \cos 2t + 5 \sin 2t$$

When $t = 0.2$, $$\frac{dv}{dt} = 10(0.2)\cos 2(0.2) + 5\sin 2(0.2)$$
$= 2\cos 0.4 + 5 \sin 0.4$ (where $\cos 0.4$ means the cosine of 0.4 radians)

Hence $$\frac{dv}{dt} = 2(0.92106) + 5(0.38942)$$
$$= 1.8421 + 1.9471 = 3.7892$$

i.e., the rate of change of voltage when $t = 0.2$ s is 3.79 volts/s, correct to 3 significant figures.

Exercise 2. Differentiation of a product

E. Differentiation of a Quotient

When $y = \frac{u}{v}$, and u and v are both functions of x

then $$\boldsymbol{\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}}$$

This is known as the **quotient rule**.

Problem 14. Find the differential coefficient of $y = \frac{4\sin 5x}{5x^4}$.

$\frac{4\sin 5x}{5x^4}$ is a quotient. Let $u = 4\sin 5x$ and $v = 5x^4$

(Note that v is **always** the denominator and u the numerator)

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

where $\frac{du}{dx} = (4)(5)\cos 5x = 20\cos 5x$

and $\frac{dv}{dx} = (5)(4)x^3 = 20x^3$

Hence $$\frac{dy}{dx} = \frac{(5x^4)(20\cos 5x) - (4\sin 5x)(20x^3)}{(5x^4)^2}$$
$$= \frac{100x^4\cos 5x - 80x^3\sin 5x}{25x^8}$$
$$= \frac{20x^3[5x\cos 5x - 4\sin 5x]}{25x^8}$$

i.e. $$\boldsymbol{\frac{dy}{dx} = \frac{4}{5x^5}(5x\cos 5x - 4\sin 5x)}$$

Note that the differential coefficient is **not** obtained by merely differentiating each term in turn and then dividing the numerator by the denominator. The quotient formula **must** be used when differentiating quotients.

Problem 15. Determine the differential coefficient of $y = \tan ax$.

$y = \tan ax = \frac{\sin ax}{\cos ax}$. Differentiation of $\tan ax$ is thus treated as a quotient with $u = \sin ax$ and $v = \cos ax$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$
$$= \frac{(\cos ax)(a\cos ax) - (\sin ax)(-a\sin ax)}{(\cos ax)^2}$$
$$= \frac{a\cos^2 ax + a\sin^2 ax}{(\cos ax)^2} = \frac{a(\cos^2 ax + \sin^2 ax)}{\cos^2 ax}$$
$$= \frac{a}{\cos^2 ax}, \text{ since } \cos^2 ax + \sin^2 ax = 1$$

(see Chapter 16)

Hence $\boldsymbol{\frac{dy}{dx} = a\sec^2 ax}$ since $\sec^2 ax = \frac{1}{\cos^2 ax}$

Problem 16. Find the derivative of $y = \sec ax$.

$y = \sec ax = \frac{1}{\cos ax}$ (i.e. a quotient). Let $u = 1$ and $v = \cos ax$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$
$$= \frac{(\cos ax)(0) - (1)(-a\sin ax)}{(\cos ax)^2}$$
$$= \frac{a\sin ax}{\cos^2 ax} = a\left(\frac{1}{\cos ax}\right)\left(\frac{\sin ax}{\cos ax}\right)$$

i.e. $$\boldsymbol{\frac{dy}{dx} = a\sec ax\tan ax}$$

Problem 17. Differentiate $y = \dfrac{te^{2t}}{2\cos t}$

The function $\dfrac{te^{2t}}{2\cos t}$ is a quotient, whose numerator is a product.

Let $u = te^{2t}$ and $v = 2\cos t$ then

$\dfrac{du}{dt} = (t)(2e^{2t}) + (e^{2t})(1)$ and $\dfrac{dv}{dt} = -2\sin t$

$$\text{Hence } \frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$= \frac{(2\cos t)[2te^{2t} + e^{2t}] - (te^{2t})(-2\sin t)}{(2\cos t)^2}$$

$$= \frac{4te^{2t}\cos t + 2e^{2t}\cos t + 2te^{2t}\sin t}{4\cos^2 t}$$

$$= \frac{2e^{2t}[2t\cos t + \cos t + t\sin t]}{4\cos^2 t}$$

i.e. $$\mathbf{\frac{dy}{dx} = \frac{e^{2t}}{2\cos^2 t}(2t\cos t + \cos t + t\sin t)}$$

Problem 18. Determine the gradient of the curve $y = \dfrac{5x}{2x^2 + 4}$ at the point $\left(\sqrt{3}, \dfrac{\sqrt{3}}{2}\right)$.

Let $y = 5x$ and $v = 2x^2 + 4$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} = \frac{(2x^2+4)(5) - (5x)(4x)}{(2x^2+4)^2}$$

$$= \frac{10x^2 + 20 - 20x^2}{(2x^2+4)^2} = \frac{20 - 10x^2}{(2x^2+4)^2}$$

At the point $\left(\sqrt{3}, \dfrac{\sqrt{3}}{2}\right)$, $x = \sqrt{3}$,

$$\text{hence the gradient} = \frac{dy}{dx} = \frac{20 - 10(\sqrt{3})^2}{[2(\sqrt{3})^2 + 4]^2}$$

$$= \frac{20 - 30}{100} = \mathbf{-\frac{1}{10}}$$

Exercise 3. Differentiation of a quotient.

F. Function of a Function

It is often easier to make a substitution before differentiating.

If y is a function of x then $\boxed{\dfrac{\mathbf{dy}}{\mathbf{dx}} = \dfrac{\mathbf{dy}}{\mathbf{du}} \times \dfrac{\mathbf{du}}{\mathbf{dx}}}$

This is known as the **'function of a function'** rule (or sometimes the **chain rule**).

For example, if $y=(3x-1)^9$ then, by making the substitution $u=(3x-1)$, $y=u^9$, which is of the 'standard' form.

Hence $\dfrac{dy}{du}=9u^8$ and $\dfrac{du}{dx}=3$

Then $\dfrac{dy}{dx}=\dfrac{dy}{du}\times\dfrac{du}{dx}=(9u^8)(3)=27u^8$

Rewriting u as $(3x-1)$ gives: $\mathbf{\dfrac{dy}{dx}=27(3x-1)^8}$

Since y is a function of u, and u is a function of x, then y is a function of a function of x.

Problem 19. Differentiate $y=3\cos(5x^2+2)$.

Let $u=5x^2+2$ then $y=3\cos u$

Hence $\dfrac{du}{dx}=10x$ and $\dfrac{dy}{du}=-3\sin u$.

Using the function of a function rule,

$$\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}=(-3\sin u)(10x)=-30x\sin u$$

Rewriting u as $5x^2+2$ gives:

$$\mathbf{\frac{dy}{dx}=-30x\sin(5x^2+2)}$$

Problem 20. Find the derivative of $y=(4t^3-3t)^6$.

Let $u=4t^3-3t$, then $y=u^6$

Hence $\dfrac{du}{dt}=12t^2-3$ and $\dfrac{dy}{du}=6u^5$

Using the function of a function rule,

$$\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}=(6u^5)(12t^2-3)$$

Rewriting u as $(4t^3-3t)$ gives:

$$\begin{aligned}\mathbf{\frac{dy}{dt}} &= 6(4t^3-3t)^5(12t^2-3)\\ &= \mathbf{18(4t^2-1)(4t^3-3t)^5}\end{aligned}$$

Problem 21. Determine the differential coefficient of $y=\sqrt{(3x^2+4x-1)}$.

$y=\sqrt{(3x^2+4x-1)}=(3x^2+4x-1)^{\frac{1}{2}}$

Let $u=3x^2+4x-1$ then $y=u^{\frac{1}{2}}$

Hence $\dfrac{du}{dx}=6x+4$ and $\dfrac{dy}{du}=\dfrac{1}{2}u^{-\frac{1}{2}}=\dfrac{1}{2\sqrt{u}}$

Using the function of a function rule,

$$\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}=\left(\frac{1}{2\sqrt{u}}\right)(6x+4)=\frac{3x+2}{\sqrt{u}}$$

i.e. $\mathbf{\dfrac{dy}{dx}=\dfrac{3x+2}{\sqrt{(3x^2+4x-1)}}}$

Problem 22. Differentiate $y=3\tan^4 3x$.

Let $u=\tan 3x$ then $y=3u^4$

Hence $\dfrac{du}{dx}=3\sec^2 3x$, (from Problem 15), and

$\dfrac{dy}{du}=12u^3$

Then $\dfrac{dy}{dx}=\dfrac{dy}{du}\times\dfrac{du}{dx}=(12u^3)(3\sec^2 3x)$

$=12(\tan 3x)^3(3\sec^2 3x)$

i.e. $\mathbf{\dfrac{dy}{dx}=36\tan^3 3x\sec^2 3x}$

Problem 23. Find the differential coefficient of $y=\dfrac{2}{(2t^3-5)^4}$

$y = \dfrac{2}{(2t^3 - 5)^4} = 2(2t^3 - 5)^{-4}$. Let $u = (2t^3 - 5)$,
then $y = 2u^{-4}$

Hence $\dfrac{du}{dt} = 6t^2$ and $\dfrac{dy}{du} = -8u^{-5} = \dfrac{-8}{u^5}$

Then $\dfrac{dy}{dt} = \dfrac{dy}{du} \times \dfrac{du}{dt} = \left(\dfrac{-8}{u^5}\right)(6t^2)$

$$= \mathbf{\frac{-48t^2}{(2t^3 - 5)^5}}$$

Exercise 4. Function of a function

G. Successive Differentiation

When a function $y = f(x)$ is differentiated with respect to x the differential coefficient is written as $\frac{dy}{dx}$ or $f'(x)$. If the expression is differentiated again, the second differential coefficient is obtained and is written as $\frac{d^2y}{dx^2}$ (pronounced dee two y by dee x squared) or $f''(x)$ (pronounced f double-dash x).

By successive differentiation further higher derivatives such as $\frac{d^3y}{dx^3}$ and $\frac{d^4y}{dx^4}$ may be obtained.

Thus if $y = 3x^4$, $\frac{dy}{dx} = 12x^3$, $\frac{d^2y}{dx^2} = 36x^2$,

$\frac{d^3y}{dx^3} = 72x$, $\frac{d^4y}{dx^4} = 72$ and $\frac{d^5y}{dx^5} = 0$.

Problem 24. If $f(x) = 2x^5 - 4x^3 + 3x - 5$, find $f''(x)$.

$$f(x) = 2x^5 - 4x^3 + 3x - 5$$
$$f'(x) = 10x^4 - 12x^2 + 3$$
$$\boldsymbol{f''(x)} = 40x^3 - 24x = \mathbf{4x(10x^2 - 6)}$$

Problem 25. If $y = \cos x - \sin x$, evaluate x, in the range $0 \le x \le \frac{\pi}{2}$, when $\frac{d^2y}{dx^2}$ is zero.

Since $y = \cos x - \sin x$, $\frac{dy}{dx} = -\sin x - \cos x$ and $\frac{d^2y}{dx^2} = -\cos x + \sin x$.

When $\frac{d^2y}{dx^2}$ is zero, $-\cos x + \sin x = 0$,

i.e. $\sin x = \cos x$ or $\frac{\sin x}{\cos x} = 1$.

Hence $\tan x = 1$ and $\boldsymbol{x} = \arctan 1 = \mathbf{45°}$ **or** $\boldsymbol{\frac{\pi}{4}}$ **rads** in the range $0 \le x \le \frac{\pi}{2}$

Problem 26. Given $y = 2xe^{-3x}$ show that

$$\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 9y = 0.$$

$y = 2xe^{-3x}$ (i.e. a product)

Hence
$$\frac{dy}{dx} = (2x)(-3e^{-3x}) + (e^{-3x})(2)$$
$$= -6xe^{-3x} + 2e^{-3x}$$
$$\frac{d^2y}{dx^2} = [(-6x)(-3e^{-3x}) + (e^{-3x})(-6)] + (-6e^{-3x})$$
$$= 18xe^{-3x} - 6e^{-3x} - 6e^{-3x}$$

i.e. $$\frac{d^2y}{dx^2} = 18xe^{-3x} - 12e^{-3x}$$

Substituting values into $\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 9y$ gives:

$$(18xe^{-3x} - 12e^{-3x}) + 6(-6xe^{-3x} + 2e^{-3x}) + 9(2xe^{-3x}) = 18xe^{-3x} - 12e^{-3x} - 36xe^{-3x} + 12e^{-3x} + 18xe^{-3x} = 0$$

Thus when $y = 2xe^{-3x}$, $\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 9y = 0$

Problem 27. Evaluate $\frac{d^2y}{d\theta^2}$ when $\theta = 0$ given $y = 4 \sec 2\theta$.

Since $y = 4 \sec 2\theta$,

then
$$\frac{dy}{d\theta} = (4)(2) \sec 2\theta \tan 2\theta \text{ (from Problem 16)}$$
$$= 8 \sec 2\theta \tan 2\theta \text{ (i.e. a product)}$$
$$\frac{d^2y}{d\theta^2} = (8 \sec 2\theta)(2 \sec^2 2\theta) + (\tan 2\theta)[(8)(2) \sec 2\theta \tan 2\theta]$$
$$= 16 \sec^3 2\theta + 16 \sec 2\theta \tan^2 2\theta$$

When $\theta = 0$, $$\frac{d^2y}{d\theta^2} = 16 \sec^3 0 + 16 \sec 0 \tan^2 0$$
$$= 16(1) + 16(1)(0) = \mathbf{16}.$$

Exercise 5. Successive differentiation